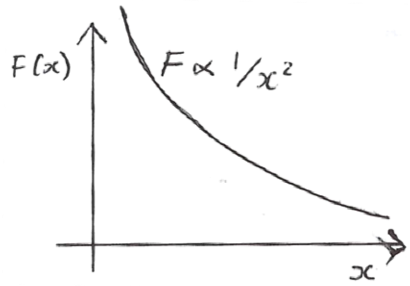
Solutions on charge distributions

Throughout these solutions the constant $\frac{1}{4\pi\epsilon_0}$ will be written as k for the working and in full for the final answers

Questions and solutions

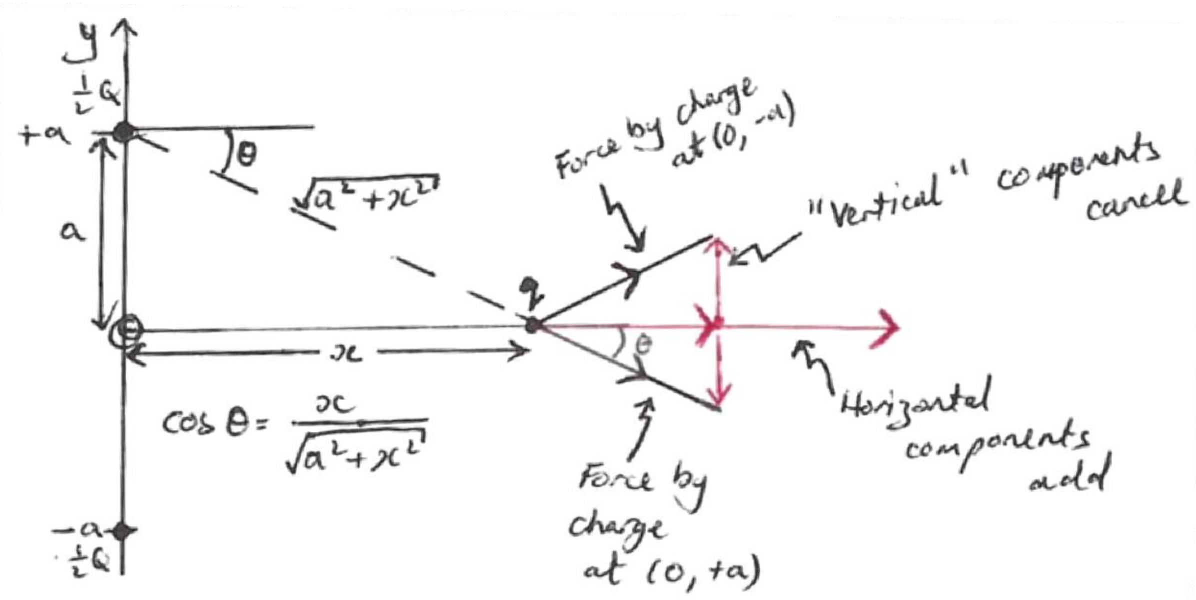
1. To start with, if a point charge $+Q$ is placed at the origin write down the magnitude of force on the test charge as a function of x and sketch a graph of force vs. distance.

   This is just Coulomb's law for point charges, so the magnitude of the force is given by $F(x) = \frac{1}{4\pi\epsilon_0}\frac{Qq}{x^2}$ and the plot is as shown below:



2. Now consider two point charges of $+\frac{1}{2}Q$ each positioned on the y-axis at $y = +a$ and $y = -a$. What would you expect the force on the test charge to be at (a) $x = 0$ and (b) when $x \gg a$? Use Coulomb's law and the principle of superposition to find an expression for the force as a function of x. Does the expression confirm your initial guesses? Sketch a graph of force vs. distance, and show that the force is a maximum at $x = \frac{1}{\sqrt{2}}a$.

   Sketching a diagram of the situation indicating the relevant geometry and forces:



Directly between the two charges the force should be zero, and far away the two charges should act as a single point charge of value $+Q$.

The two charges both exert a force on the test charge. Using Coulomb's law and Pythagoras' theorem these forces both have a magnitude of

$$\frac{k\frac{1}{2}Qq}{a^2 + x^2}$$

and according to the principle of superposition they add as vectors.

If these forces are split into perpendicular components aligned along the x- and y- axes it can be seen from the geometry that the y- components are equal and opposite and therefore cancel and the x- components are equal and additive.

Each force's x- component has a value of

$$\frac{k\frac{1}{2}Qq}{a^2 + x^2}\cos\theta$$

with the angle as indicated on the diagram, to give total force on the test charge of

$$F(x) = 2\frac{k\frac{1}{2}Qq}{a^2 + x^2}\cdot\frac{x}{\sqrt{a^2 + x^2}} \equiv \frac{Qq}{4\pi\epsilon_0}\cdot\frac{x}{(a^2 + x^2)^{3/2}}$$

The expression for $F(x)$ confirms that the force is zero when the test charge is at the origin and when $x \gg a$ then $(a^2 + x^2)^{3/2} \approx x^3$ i.e. the expression approaches Coulomb's law at large distances as suggested.

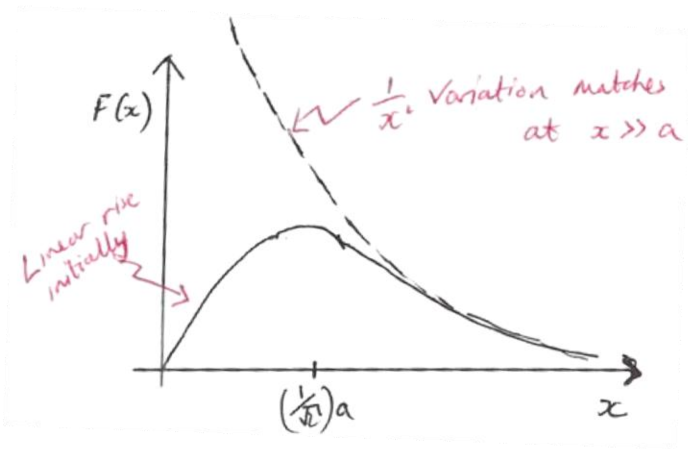
Though not a question requirement a little more thought shows that for small distances from the origin $F(x) \approx \frac{kQq}{a^3}x$ i.e. the force is directly proportional to the displacement; for unlike charges this would mean the test charge would oscillate about the origin with SHM for small displacements.

The derivative of the function is

$$\frac{dF}{dx} = kQq\left\{\frac{(a^2 + x^2)^{3/2} - 3x^2(a^2 + x^2)^{1/2}}{(a^2 + x^2)^3}\right\}$$

(yuck) which has solutions when set to zero of $x = \mp\frac{1}{\sqrt{2}}a$ as required.

The resulting graph of force vs. distance is as below:



3. (a) Now consider four point charges each of value $+\frac{1}{4}Q$. Two of them are placed in the same positions as before on the y-axis at $y = +a$ and $y = -a$. The other two are placed on the z-axis at $z = +a$ and $z = -a$. Find the force on the test charge as a function of x as before. NB no further detailed calculation is required to do this though it may help to start setting up the calculation in the same way as before to begin.

In this case each charge exerts a force that is half of the magnitude of the force due to each point charge in question 2. As before the y- components of the forces will cancel, and in this case so will the z- components. The x- components are all aligned again, and though each of them contributes half as much as in question 2, there are twice as many so the force is identical to the expression derived in 2.

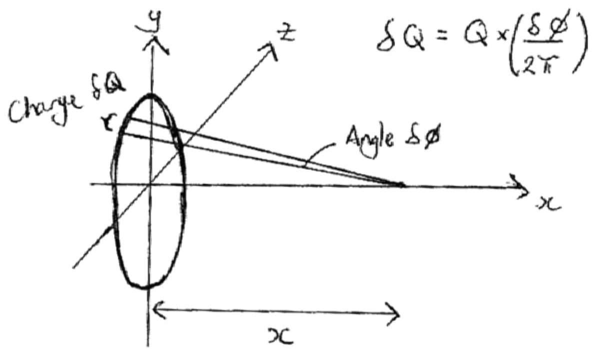
(b) A large number, N, of equally spaced point charges of value $+\frac{1}{N}Q$ are arranged in a circle of radius a on the yz plane with the centre of the circle at the origin. What is the force on the test charge as a function of x in this case?

For this extension of the situation should be the same again: all the y- and z- force components from the charges will cancel out and the x- components will add. Each gives $\frac{2}{N}$th of the effect of one of the charges in question 2 but as there $\frac{N}{2}$ times as many the overall effect is the same.

(c) Parts (a) and (b) referred to <u>discrete</u> charge distributions. Now consider a <u>continuous</u> charge distribution: an infinitesimally thin circular hoop of radius a with total charge $+Q$ and a uniform charge distribution is placed in the yz plane with its centre at the origin. Use the answers from the previous questions to guess the expression for the force on the test charge as a function of x. Using the principle of superposition and integral calculus to perform a calculation to confirm or refute your guess.

As a logical extension of the previous question, as $N \to \infty$ the more the circle of discrete charges approaches a uniform distribution so the effect should be the same again.

To show this more rigorously, consider the diagram below:



The diagram shows an element of charge, δQ, that subtends an angle $\delta\phi$ as shown. As the charge on the hoop is uniformly distributed its value is thus $\delta Q = Q\frac{\delta\phi}{2\pi}$.

This element of charge exerts an element of force component in x- direction of

$$\delta F_x = \frac{kQq}{a^2+x^2}\cdot\frac{x}{\sqrt{a^2+x^2}}\cdot\frac{\delta\phi}{2\pi}$$

To get the total force, the contributions must be summed over the whole angle range i.e. integrated from $\phi = 0 \to 2\pi$. This gives

$$F_x = \frac{kQq}{a^2+x^2}\cdot\frac{x}{\sqrt{a^2+x^2}}\int_0^{2\pi}\frac{d\phi}{2\pi} = \frac{kQq}{a^2+x^2}\cdot\frac{x}{\sqrt{a^2+x^2}}$$

as expected.

4. An infinitesimally thin, solid disc of radius a and total charge $+Q$ with a uniform charge distribution is arranged in the same way: in the yz plane with its centre at the origin. Use integral calculus to find show that the force as a function of x is given by $F(x) = \frac{1}{4\pi\epsilon_0}\frac{2Qq}{a^2}\left(1-\frac{x}{\sqrt{x^2+a^2}}\right)$.

   Use the expression to determine the nature of the force close to and far away from the disc and sketch a graph of $F(x)$ vs. x. For the case when $x \gg a$ use of the binomial expansion can help reach a useful approximation. (NB the integral that is needed may look awkward at first but can be guessed at by inspection – or just looked up if necessary.)

   The disc can be thought of as a series of concentric hoops of thickness δr.

   Each of these hoops carries a charge of

$$Q \times \frac{2\pi r.\delta r}{\pi a^2} = \frac{2Qr.\delta r}{a^2}$$

   So each of these hoops makes an elemental force contribution of

$$\frac{2kQqxr.\delta r}{a^2(r^2+x^2)^{3/2}}$$

The total force contribution is given by the integral of this from $r = 0 \rightarrow a$ i.e.

$$F(x) = \frac{2kQqx}{a^2}\int_0^a \frac{r.dr}{(r^2 + x^2)^{3/2}}$$

The integral looks complicated but the indefinite solution is simply $-\frac{1}{(r^2+x^2)^{1/2}}$ which when plugged in after a little algebra renders the expression required.

Close to the disc, when $a \gg x$ the expression becomes

$$F(x) = \frac{1}{4\pi\epsilon_0}\frac{2Qq}{a^2}\left(1 - \frac{x}{a}\right)$$

i.e. the maximum value is $\frac{1}{4\pi\epsilon_0}\frac{2Qq}{a^2}$ and then the force drops mildly and linearly.

Beyond there the force monotonically decreases, and for large distance it once again approximates that of a point charge as when $x \gg a$ then $\frac{x}{\sqrt{x^2+a^2}} \approx 1 - \frac{a^2}{2x^2}$ from the binomial expansion.

The plot of force vs. distance is thus: