# Vibrations and Waves – Tutorial Questions VnW1 ANSWERS

## ANSWER TO QUESTION

(i) $[k] = [\tau/\theta] = \mathrm{N\,m/rad}$

(ii) $\tau = I\ddot{\theta}$ (Compare with $F = m\ddot{x}$.)

$-k\theta = I\ddot{\theta} \quad \Rightarrow \quad \ddot{\theta} + (k/I)\theta = 0$

(Note this is the same as a mass on a spring with $m \to I$ and $x \to \theta$.)

(iii) Differentiating, $\theta(t) = A\cos(\omega_0 t + \phi)$ twice gives $\ddot{\theta} = -\omega_0^2 A\cos(\omega_0 t + \phi)$. Substituting into the equation of motion gives

$$-\omega_0^2 A\cos(\omega_0 t + \phi) + (k/I)A\cos(\omega_0 t + \phi) = 0$$

Cancelling $A\cos(\omega_0 t + \phi)$, leaves $\omega_0^2 = k/I$ — i.e., trial solution is valid if $\omega_0^2 = k/I$. Furthermore, it has two independently adjustable constants as expected for the general solution of a 2nd-order differential equation.

Similarly for $\tilde{\theta}(t) = A\exp(\mathrm{i}\omega_0 t + \phi)$, differentiating gives $\ddot{\tilde{\theta}} = -\omega_0^2 A\exp(\omega_0 t + \phi)$. Substituting into the equation of motion and cancelling yields the same result, $\omega_0^2 = k/I$.

Hence $\omega_0 = \sqrt{k/I}$ and $T = 2\pi/\omega_0 = 2\pi\sqrt{I/k}$.

(iv) From (iii), $T = 2\pi\sqrt{I/k}$. Rearranging and substituting for $I$ gives

$$k = I\frac{4\pi^2}{T^2} = mR^2\frac{4\pi^2}{T^2} = \frac{4\pi^2 mR^2}{T^2} = \frac{4\pi^2(40\times10^{-6}\,\mathrm{kg})(4.5\times10^{-3}\,\mathrm{m})^2}{(^1/_3\mathrm{s})^2} = 2.88\times10^{-7}\,\mathrm{Nm/rad}$$

(v) (a) Initial conditions $\theta(0) = 4.5$ rad and $\dot{\theta}(0) = 0$. Sub into general solution, gives

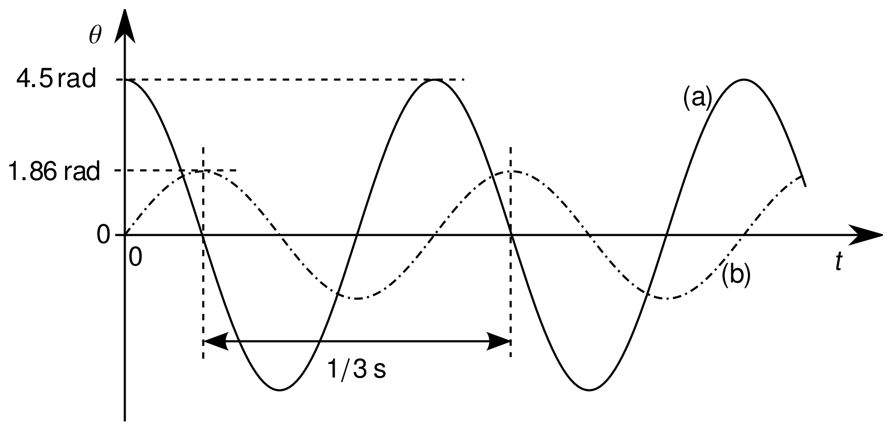
$$\theta(0) = A\cos\phi = 4.5\,\mathrm{rad}, \quad \dot{\theta}(0) = -\omega_0 A\sin\phi = 0 \quad \Rightarrow \quad A = 4.5\,\mathrm{rad}, \quad \phi = 0$$

$$\left(\text{more generally, can use:} \qquad A^2 = \theta^2 + \dot{\theta}^2/\omega_0^2, \quad \tan\phi = \frac{-\dot{\theta}(0)}{\omega_0\theta(0)}\right)$$

(b) Initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 35.0$ rad/s.

$$\theta(0) = A\cos\phi = 0, \quad \dot{\theta}(0) = -\omega_0 A\sin\phi = 35.0\,\mathrm{rad/s}$$

$$\Rightarrow \quad A = 35.0\,\mathrm{rad}/\omega_0 = \frac{35.0}{2\pi\times3} = 1.86\,\mathrm{rad}, \quad \phi = -\pi/2$$



(vi) (a) Using $x \to \theta$ and $m \to I$,

$$\mathrm{PE} = \frac{1}{2}k[x(t)]^2 \quad \to \quad \mathrm{PE}_{\mathrm{rot}} = \frac{1}{2}k[\theta(t)]^2$$

$$\mathrm{KE} = \frac{1}{2}m[v(t)]^2 \quad \to \quad \mathrm{KE}_{\mathrm{ang}} = \frac{1}{2}I[\dot{\theta}(t)]^2$$

$$\mathrm{E} = \mathrm{PE}_{\mathrm{rot}} + \mathrm{KE}_{\mathrm{ang}} = \frac{1}{2}\left(k\theta^2 + I\dot{\theta}^2\right)$$

(b) Differentiating total energy E, which is conserved, w.r.t. $t$ gives

$$k\theta\dot{\theta} + I\dot{\theta}\ddot{\theta} = 0 \quad \Rightarrow \quad k\theta + I\ddot{\theta} = 0$$

## TOPICS FOR DISCUSSION

It is essential that the students understand the following topics early in this course.

**Topic 1**

Prove that

$$x(t) = A\sin(\omega t + \theta)$$

and

$$x(t) = B\cos\omega t + C\sin\omega t$$

are both mathematically equivalent to $x(t) = A\cos(\omega t + \phi)$, where $\theta$, $B$, and $C$ are constants. Write down expressions relating $\phi$ to $\theta$, $A$ to $B$ and $C$, and $\phi$ to $B$ and $C$.

**Topic 2: Using complex variables**

See Sections 1.4 and 1.5 of my notes (on Blackboard) on circular motion and motion in the complex plane.

Possible exercise (from Section 1.5 of notes):

Show that if $\tilde{x}(t) = x(t) + iy(t)$ is a solution to the differential equation

$$a\frac{d^2\tilde{x}}{dt^2} + b\frac{d\tilde{x}}{dt} + c\tilde{x} = f(t)$$

where $a$, $b$, $c$, $x(t)$, and $y(t)$ are all real valued, then

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = \mathrm{Re}\,(f(t))$$

and

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \mathrm{Im}\,(f(t))$$

where Re() and Im() refer to the real and imaginary parts, respectively.

**Topic 3: Taylor series expansion (1-D and 2-D)**

The following question used to be on the tutorial sheet. I've removed it as a question because I've covered it in detail in notes and lectures, but it is very important that they understand Taylor series and linear response.

**Q:**

Show that the restoring force about any stable equilibrium point is linear (i.e. Hooke's Law is valid) for sufficiently small displacements. Why is this important? What happens for larger displacements?

**A:**

Taylor series expansion of $V(x)$ about $x_0$ gives

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + (1/2)V''(x_0)(x - x_0)^2 + (1/3!)V'''(x_0)(x - x_0)3 + ...$$

Because $x_0$ is a stable equilibrium, $V'(x_0) = 0$ (no net force) and $V''(x_0) > 0$ (local minimum of $V$.) Therefore the force, $F(x) = -\,\mathrm{d}V(x)/\,\mathrm{d}x$ is given by

$$F(x) = -V''(x_0)(x - x_0) - (1/2)V'''(x_0)(x - x_0)^2 - ...$$

For small displacements $(x - x_0)$ we can neglect the second and higher terms, so that $F(x) = -V''(x_0)(x - x_0) = -k(x - x_0)$, where $k = V''(x_0) > 0$ i.e., a linear restoring force.

Note, small displacements are such that $|[(1/2)V'''(x_0)/V''(x_0)](x - x_0)| << 1$ (with similar relationships for the higher order terms.)

This is an important point: physical systems are not purely linear or purely non-linear, but have ranges in which they can be treated as linear and ranges in which they are non-linear. However, all behave linearly if the displacement is small enough. This is a consequence of $V(x)$ being a smooth function of $x$.

If you allow the displacements to increase, you should see that there comes a point when the second and higher terms that you have neglected here start to become relevant again and the system starts to become non-linear. Of course the size of the linear range can vary enormously between different systems.

In a non-linear system the oscillatory motion is no longer simple harmonic and it contains components with different frequencies.