Electromagnetism solutions problem 4: the continuity equation

Section 6.2 looks at a flow of charge. Considering a flow of mass instead, in the form of, say, the flowing of a river, follow the working of the section to derive the continuity equation for mass.

This is tricky for 1st year level and is not the sort of thing that would feature on an exam for this course. It is the kind of thing that could easily be brought into the Comprehensive paper in the 3rd year however.

The following is just one approach to it.

The first thing to do is recognise what the analogous quantities to charge, current and charge density and current density are in mass flow:

The analogy to charge is simply mass, m.

And the analogy to current – rate of flow of charge – is simply the rate of flow of mass in kgs^{-1}. Rather than invent a special symbol for this it can just be written as $\frac{dm}{dt}$.

The analogy to charge density is simply the density for mass that we are used to. For non-uniform densities the best way to express the mass within a volume is by writing $m = \int_V \rho(V)\, dV$.

The analogy to current density is trickier. Consider a uniform mass flow passing through an area A when flowing at a velocity $\boldsymbol{v}$ parallel to the area vector i.e. normal to the plane of the area. The mass that passes per second is given by $\rho A v$ and the analogous equation to 1.5 is $\frac{dm}{dt} = \int_S (\rho \boldsymbol{v}).\hat{\boldsymbol{n}}.dA = \int_S (\rho \boldsymbol{v}).d\boldsymbol{S}$.

i.e. the mass flow analogy to the current density field is the "density times velocity" field which is also uniquely defined at each point in space with three components at each position in a 3D system.

After this is established the working from the section on charge can be copied almost verbatim:

The net mass flow <u>into</u> a volume can be expressed by $\frac{dm}{dt} = -\int_S (\rho \boldsymbol{v}).d\boldsymbol{S}$ where the $d\boldsymbol{S}$ term incorporates all infinitesimal $\hat{\boldsymbol{n}}.dA$ terms over the whole surface and the minus sign indicates this gives the net inward mass flow.

Using the <u>divergence theorem</u> from vector calculus, this can be written $\frac{dm}{dt} = -\int_V \left(\nabla.(\rho\boldsymbol{v})\right)dV$ i.e. the net mass rate flowing into the volume is the negative of the integral of the divergence of the density times velocity field over the whole volume.

So using the definition of density, $\frac{dm}{dt} \equiv \int_V \frac{d\rho}{dt}dV$ thus $\int_V \frac{d\rho}{dt}dV = -\int_V \left(\nabla.(\rho\boldsymbol{v})\right)dV$

For this to always be true for any volume this gives the continuity equation for mass viz.

$$\frac{d\rho}{dt} = -\nabla.(\rho\boldsymbol{v})$$

This is, in effect, statement of the conservation of mass. What it means is that the only way for the density at a point to change is for a mass to flow into or out of that point.

An important simplification of this is for the case when the density of a fluid is constant (i.e. the fluid is incompressible) which is often assumed to be the case for liquid flows. In this case the left hand side term is zero, the density term on the right can be brought outside the divergence operator and the continuity equation is

$$\nabla.\boldsymbol{v} = 0$$

which is an equation which sees widespread use in many aspects of fluid dynamics.