Friday 16th February 2018
(Answers available from Monday 26th February)

Quantum Physics Problem Sheet 3

1. Considering the localisation in space of a free-particle wavepacket, what is the minimum momentum spread a particle must have if the position spread is: (a) $1\mu m$, (b) $0.1nm$, and (c) $10^{-15}m$?

2. A particle of mass m is confined within a potential well of the form:

$$V(x) = \begin{cases} 0 & |x| < d/2 \, , \\ \infty & \text{otherwise} \, . \end{cases}$$

The (unnormalised) ground-state wavefunction is

$$\psi(x,t) = \begin{cases} \cos(\pi x/d)\, e^{-i\left(\hbar\pi^2/2md^2\right)t} & |x| < d/2 \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

   (i) A measurement is made of the particle position. Show that the probability that the measured value lies between x and $x + dx$ is independent of time. Where is the partice most likely to be found?

   (ii) Normalise $\psi(x,t)$.

   (iii) Find $\langle x \rangle$ and $\langle x^2 \rangle$. Hence obtain the rms width Δx of the probability density function.

[The following result may be used without proof: $\int_{-\pi/2}^{\pi/2} \theta^2 \cos^2\theta \, d\theta = \frac{\pi}{4}\left(\frac{\pi^2}{6} - 1\right)$]

3. Show that the de Broglie wavelength of a particle of mass m and kinetic energy $3k_BT/2$ (i.e. the mean thermal energy of a particle at temperature T) is

$$\lambda = \frac{h}{\sqrt{3mk_BT}} \, .$$

The molar volume of liquid $^4\mathrm{He}$ is 27.6 cm^3.

(a) Assuming that each atom occupies a cube of side d what is d for $^4\mathrm{He}$?

(b) What is the temperature at which $\lambda = d$?

(c) Discuss the temperature range over which the wave-like properties of the atoms in liquid $^4\mathrm{He}$ are likely to be important.

(Incidentally, liquid $^4\mathrm{He}$ becomes a superfluid, able to flow without friction, below 2.17 K. The above calculation suggests that superfluidity is almost certainly a QM phenomenon.)

4. When liquid $^4\mathrm{He}$ freezes, every atom is confined to a "box" (its lattice site in the crystal). Since liquids and solids normally have similar densities, the box size is similar to the value of d calculated in Q4. Assuming that the de Broglie wave of wavelength λ associated with the $^4\mathrm{He}$ atom has to equal zero at the box walls, show (perhaps by drawing a diagram) that $d = n\lambda/2$, where n is any integer > 0. Hence calculate the smallest possible momentum and kinetic energy of the confined atom. At what temperature T would the thermal kinetic energy $3k_BT/2$ equal the quantum mechanical KE of confinement?

Assessed Problem

5. (i) The wavefunction of a particle confined in a box is constrained to satisfy a strict set of boundary conditions. Show that for a box of width d the allowed values of the energy of a particle of mass m are quantised and take discrete values:

$$E_n = \frac{h^2 n^2}{8md^2} \; .$$

where n is an integer.

[4 marks]

(ii) For an electron in a 0.5 nm wide box what will be the minimum energy of the confined electron? [2 marks]

(iii) Treating a deep quantum well structure of width $5 \times 10^{-10} m$ confining an electron as an infinite square well, what will be the energy of a photon emitted between the third excited state (n=3) and the minimum energy state?

[2 marks]

(iv) What is the expectation value $< p >$ of the electron's momentum in the n=3 state?

[2 marks]

Tutorial Problem (or you may choose to work through 2 carefully)

6. When undergoing radioactive decay, nuclei often emit electrons with energies between 1 and 10 MeV. Use the position-momentum uncertainty principle to show that an electron of energy 1 MeV could not have been contained in the nucleus before the decay.

Physical Constants

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\begin{aligned}
m_{\mathrm{e}} &\approx 9.11 \times 10^{-31}\,\mathrm{kg} \approx 511\,\mathrm{keV}/c^2 \\
m_{\mathrm{n}} &\approx 1.67 \times 10^{-27}\,\mathrm{kg} \\
\text{atomic mass unit} &\approx 1.66 \times 10^{-27}\,\mathrm{kg} \\
h &\approx 6.63 \times 10^{-34}\,\mathrm{Js} \\
\hbar &\approx 1.05 \times 10^{-34}\,\mathrm{Js} \\
c &\approx 3.00 \times 10^{8}\,\mathrm{ms}^{-1} \\
e &\approx 1.60 \times 10^{-19}\,\mathrm{C} \\
k_B &\approx 1.38 \times 10^{-23}\,\mathrm{JK}^{-1} \\
N_{\mathrm{A}} &\approx 6.02 \times 10^{23} \\
\mathrm{R_H} &\approx 1.097 \times 10^{7}\,\mathrm{m}^{-1} \\
\epsilon_0 &\approx 8.85 \times 10^{-12}\,\mathrm{C^2 J^{-1} m^{-1}}
\end{aligned}
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