Friday 23rd February 2018
(Answers available on Monday 5th March)

Quantum Physics Problem Sheet 4

1. Light of frequency 7.316×10^{14} Hz is emitted in a downward transition to the $n = 2$ energy level of a hydrogen atom. What was the initial energy level?

2. Generalise the derivation of the Bohr model given in lectures to obtain a formula for the energy levels of ions such as He^+ or Al^{12+}, which have atomic number $Z > 1$ ($Z = 2$ for He; $Z = 13$ for Al) but only one orbiting electron.

3. (i) What is the *shortest* wavelength photon that could be emitted by a hydrogen atom if it ends up in the $n = 1$ level?

   (ii) What is the *shortest* wavelength photon that could be emitted by a hydrogen atom that did not end up in the ground state $n = 1$ level?

   (iii) A spectral line at 121.5 nm is observed in both the H atom emission spectrum and, at exactly the same wavelength, in the spectrum from the singly ionized He^+ ion. What are the initial and final (n_i and n_f) levels of this transition in the H and He^+ spectra?

4. A particle of mass m is bound in a spherical potential of the form $V(r) = Cr$. By adapting the derivation of the Bohr model to this case, show that:

   (i) the quantised orbital radii are

   $$r_n = \left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} \qquad (n \text{ any integer} \geq 1)\ ;$$

   (ii) the quantised energy levels are

   $$E_n = \frac{3}{2}\left(\frac{C^2\hbar^2 n^2}{m}\right)^{1/3} \qquad (n \text{ any integer} \geq 1)\ .$$

Assessed Problem

5. The Bohr model is usually applied by considering an electron orbiting a nucleus of infinite mass. It can easily be generalised to apply to both particles having finite mass. To do this we need to replace the electron mass with the system reduced mass and then proceed as before.

   (i) For a system comprising a negatively charged particle of mass m_1 orbiting a positively charged particle of mass m_2 and interacting via the Coulomb interaction what will be the energy of the state with quantum number n according to the Bohr model?

   (a)

$$E_{\text{binding}} = \frac{m_1 e^4}{2(4\pi\epsilon_0\hbar)^2} .$$

   (b)

$$E_{\text{binding}} = \frac{\frac{m_1 m_2}{(m_1+m_2)} e^4}{2(4\pi\epsilon_0\hbar)^2} .$$

   (c)

$$E_{\text{binding}} = \frac{\frac{m_1}{(m_2)} e^4}{2(4\pi\epsilon_0\hbar)^2} .$$

   (d)

$$E_{\text{binding}} = \frac{\frac{m_2}{(m_1)} e^4}{2(4\pi\epsilon_0\hbar)^2} .$$

   [2 marks]

   (ii) An electron and a positron (same mass, opposite charge) can form a short-lived bound state called a positronium atom, in which the two particles orbit their centre of mass. Using the Bohr model what is the binding energy of the ground state of this system in eV?

   [2 marks]

   (iii) What is the distance between the electron and the positron in the positronium ground state in Angstrom?

   [2 marks]

   (iv) What is the ratio of the lowest energy state of a muonium atom (this comprises an electron orbiting a positively charged muon, where a muon mass is $207m_e$) to that of a H atom?

   [4 marks]

Tutorial Problem (Question 2 is a very good alternative)

6. (i) Consider a QM simple harmonic oscillator of mass m, spring constant s and natural frequency $\omega = \sqrt{s/m}$. Assuming that the average displacement $\langle x \rangle$ and average momentum $\langle p \rangle$ are both zero, show that the mean energy,

$$\langle E \rangle \;=\; \langle KE \rangle + \langle PE \rangle \;=\; \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} s \langle x^2 \rangle \; ,$$

may be expressed in the form:

$$\langle E \rangle \;=\; \frac{(\Delta p)^2}{2m} + \frac{1}{2} m \omega^2 (\Delta x)^2 \; ,$$

where Δx is the rms displacement and Δp is the rms momentum.

(ii) Use the uncertainty principle to show that the ground-state energy of the oscillator must be greater than or equal to $\frac{1}{2}\hbar\omega$. (In fact, the ground-state energy is *exactly* $\frac{1}{2}\hbar\omega$.)

Physical Constants

$$
\begin{aligned}
m_{\mathrm{e}} &\approx 9.11 \times 10^{-31}\,\mathrm{kg} \approx 511\,\mathrm{keV}/c^2 \\
m_{\mathrm{n}} &\approx 1.67 \times 10^{-27}\,\mathrm{kg} \\
\text{atomic mass unit} &\approx 1.66 \times 10^{-27}\,\mathrm{kg} \\
h &\approx 6.63 \times 10^{-34}\,\mathrm{Js} \\
\hbar &\approx 1.05 \times 10^{-34}\,\mathrm{Js} \\
c &\approx 3.00 \times 10^{8}\,\mathrm{ms}^{-1} \\
e &\approx 1.60 \times 10^{-19}\,\mathrm{C} \\
k_B &\approx 1.38 \times 10^{-23}\,\mathrm{JK}^{-1} \\
N_{\mathrm{A}} &\approx 6.02 \times 10^{23} \\
\mathrm{R_H} &\approx 1.097 \times 10^{7}\,\mathrm{m}^{-1} \\
\epsilon_0 &\approx 8.85 \times 10^{-12}\,\mathrm{C}^2\mathrm{J}^{-1}\mathrm{m}^{-1}
\end{aligned}
$$