Quantum Physics Answer Sheet 3

1. We will use the form of the position-momentum uncertainty relations discussed in lectures which relates the spread (uncertainty) in position Δx and momentum Δp_x through the inequality $\Delta x \Delta p_x \geq \hbar/2$.

   (a) $\Delta p_x \geq 10^{-34}/2 \times 10^{-6} \approx 5 \times 10^{-29} kgms^{-1}$.

   (b) $\Delta p_x \geq 10^{-34}/2 \times 10^{-10} \approx 5 \times 10^{-25} kgms^{-1}$.

   (c) $\Delta p_x \geq 10^{-34}/2 \times 10^{-15} \approx 5 \times 10^{-20} kgms^{-1}$.

   Of course the momentum spread grows as the position spread decreases.

2. (i) The probability density $f(x,t)$ is proportional to $|\psi(x,t)|^2$. Hence, the probability $f(x,t)dx$ that the particle is found between x and $x+dx$ at time t is proportional to

$$|\psi(x,t)|^2\,dx = \begin{cases} \cos^2(\pi x/d) e^{-i(\hbar\pi^2/2md^2)t} e^{i(\hbar\pi^2/2md^2)t}\,dx & |x| < d/2 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \cos^2(\pi x/d)\,dx & |x| < d/2 \\ 0 & \text{otherwise} \end{cases}$$

   which is independent of time. The particle is most likely to be found at $x = 0$, where $f(x)$ is largest.

   (ii) In order to normalise $\psi(x,t)$ we have to evaluate

$$N = \int_{-\infty}^{\infty} |\psi(x,t)|^2\,dx = \int_{-d/2}^{d/2} \cos^2(\pi x/d)\,dx = \frac{d}{2}\,.$$

   (The integration is easy because $\cos^2\theta$ repeats every π radians and averages to $1/2$ over any whole number of repeats.)

   Hence, the normalised ground-state wavefunction is:

$$\psi(x,t) = \begin{cases} \sqrt{\dfrac{2}{d}} \cos(\pi x/d) e^{-i(\hbar\pi^2/2md^2)t} & |x| < d/2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   (iii)

$$\langle x \rangle = \int_{-d/2}^{d/2} x\,\frac{2}{d}\cos^2\left(\frac{\pi x}{d}\right)\,dx = 0 \qquad \text{(integrand is odd)}.$$
$$\langle x^2 \rangle = \int_{-d/2}^{d/2} x^2\,\frac{2}{d}\cos^2\left(\frac{\pi x}{d}\right)\,dx$$
$$= \frac{d^3}{\pi^3}\frac{2}{d}\int_{-\pi/2}^{\pi/2} \theta^2 \cos^2\theta\,d\theta \qquad \text{(where } \theta = \pi x/d\text{)}$$
$$= \left(\frac{1}{12} - \frac{1}{2\pi^2}\right) d^2 \qquad \text{(using integral given in question).}$$

Hence

$$\Delta x \;=\; \sqrt{\langle (x - \langle x \rangle)^2 \rangle} \;=\; \sqrt{\langle x^2 \rangle} \;=\; d\sqrt{\frac{1}{12} - \frac{1}{2\pi^2}} \;\approx\; 0.181\, d\,.$$

3. A particle of mass m and momentum p has kinetic energy $p^2/2m$. If the kinetic energy is equal to $3k_BT/2$:

$$\frac{p^2}{2m} \;=\; \frac{3k_BT}{2}\,,$$

then

$$p \;=\; \sqrt{3mk_BT}\,.$$

Combining this result with de Broglie's equation, $p = h/\lambda$, gives:

$$\lambda \;=\; \frac{h}{p} \;=\; \frac{h}{\sqrt{3mk_BT}}\,,$$

as required.

Avogadro's number of He atoms occupy a volume of $27.6 \times 10^{-6}\,\mathrm{m}^3$. Hence, the volume per atom d^3 is

$$\frac{27.6 \times 10^{-6}}{6.02 \times 10^{23}} \;\approx\; 4.58 \times 10^{-29}\,\mathrm{m}^3\,.$$

Taking the cube root, we obtain $d \approx 3.58 \times 10^{-10}$ m.

To find the temperature T at which $\lambda = d$, we have to solve the equation

$$\frac{h}{\sqrt{3mk_BT}} \;=\; d\,.$$

Hence

$$T \;=\; \frac{1}{3mk_B}\left(\frac{h}{d}\right)^2 \;\approx\; \frac{1}{3 \times 4 \times 1.66 \times 10^{-27} \times 1.38 \times 10^{-23}}\left(\frac{6.63 \times 10^{-34}}{3.58 \times 10^{-10}}\right)^2$$

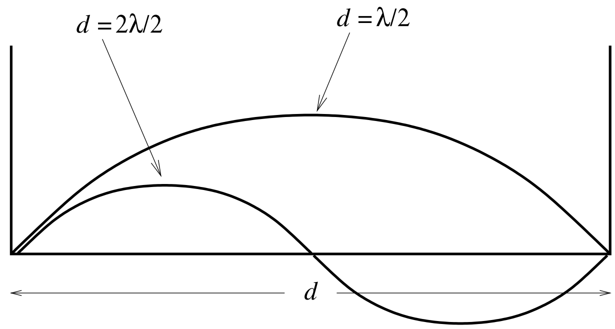
$$\approx\; 12.5\,\mathrm{K}\,.$$

When the temperature is comparable to or smaller than this value, the de Broglie wavelength of the He atoms will be the same as or greater than the interparticle spacing, and the wave-like properties of the atoms will be important.

4. The figure below shows that in order for the de Broglie wave of wavelength λ to "fit in" to the box, the box side d must be an integer multiple of $\lambda/2$: $d = n\lambda/2$, where $n = 1, 2, \ldots$

The maximum possible de Broglie wavelength is therefore $2d$. The smallest possible momentum is

$$p_{\min} \;=\; \frac{h}{\lambda_{\max}} \;=\; \frac{h}{2d} \;\approx\; \frac{6.63 \times 10^{-34}}{2 \times 3.58 \times 10^{-10}} \;\approx\; 9.26 \times 10^{-25}\,\mathrm{kg\,ms^{-1}}\,.$$



The smallest possible KE is

$$\mathrm{KE}_{\mathrm{min}} = \frac{p_{\mathrm{min}}^2}{2m} \approx \frac{(9.26\times10^{-25})^2}{2\times4\times1.66\times10^{-27}} \approx 6.46\times10^{-23}\,\mathrm{J}\ .$$

The thermal KE of $3k_BT/2$ would equal $\mathrm{KE}_{\mathrm{min}}$ when

$$T = \frac{2\,\mathrm{KE}_{\mathrm{min}}}{3k_B} \approx \frac{2\times6.46\times10^{-23}}{3\times1.38\times10^{-23}} \approx 3.12\,\mathrm{K}\ .$$

Note: The idea behind this question is important and very general. Since the de Broglie wavelength of a confined particle has to "fit in" to the confining box, the particle must have a non-zero momentum and KE. The particle must therefore be moving — rattling backwards and forwards inside the box — even at zero temperature. This confinement-induced motion is called zero-point motion, and the corresponding kinetic energy is called zero-point energy. The orbits of electrons in atoms can be viewed as a type of zero-point motion.

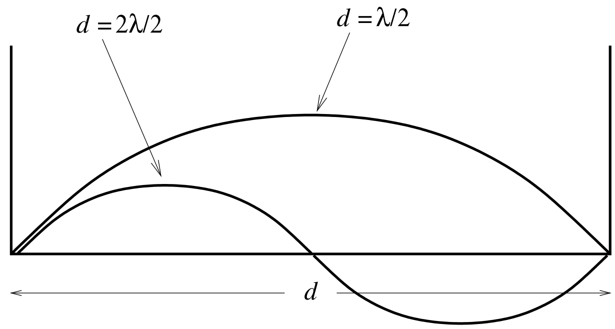
In most solids, the atoms are heavy enough and the chemical bonds strong enough that the zero-point motion of the atoms (as opposed to the electrons) is unimportant. In $^4\mathrm{He}$, however, where the atoms are comparatively light and the bonding is very weak, the zero-point motion alone is sufficient to melt the solid — there is no need to heat it up! This is why liquid $^4\mathrm{He}$ remains liquid right down to $T = 0\,\mathrm{K}$. To solidify $^4\mathrm{He}$, it is necessary to apply pressure.

5. (i) The particle is confined in the box, so the figure below shows that in order for the de Broglie wave of wavelength λ to "fit in" to the box, the box width d must be an integer multiple of $\lambda/2$: $d = n\lambda/2$, where $n = 1, 2, \ldots$

   This means that $\lambda_{dB} = 2d/n$ which then constrains the value of momentum ($p = h/\lambda$) to be $p_n = nh/2d$ (this is the constraint on the absolute value of p i.e. $|p|$, it can take + or - this value as nothing in the box constrains the direction of the momentum).

   Inside the box we assume the potential energy of the particle is zero, so the total energy is solely the kinetic energy given by: $E = p^2/2m$ so we obtain:

$$E_n = \frac{h^2n^2}{8md^2}\ .$$



[4 marks]

(ii) The minimum energy state corresponds to n=1 so:

$$E_1 = \frac{h^2 1^2}{8md^2} = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (5 \times 10^{-10})} = 2.41 \times 10^{-19} J; .$$

This corresponds to about 1.51 eV.

[2 marks]

(iii) The example has the same dimensions as the well in part (ii) so for an electron the energy is again $E_1 = 2.41 \times 10^{-19} J$. The energy of the $n = 3$ state is going to be $E_3 = 3^2 E_1$ so the energy difference between the states (and hence the energy of the photon) is $\Delta E = 8 \times 2.41 \times 10^{-19} = 1.928 \times 10^{-18} J$.

From the energy of the photon we can compute the wavelength as $\lambda = ch/E = 3.00 \times 10^{-8} \times 6.63 \times 10^{-34} / 1.93 \times 10^{-18} = 1.03 \times 10^{-7} m$

[2 marks]

(iv) See the note in part (i) (based on a comment made in lectures), although the energy is finite as $< p^2 >$ is finite, the particle is just as probable to be going to the right as to the left and so $< p >= 0$.

[2 marks]

6. The size of a nucleus is about 10^{-15} m and so the position uncertainty Δx of an electron contained in a nucleus is also about 10^{-15} m. According to the uncertainty principle, the momentum uncertainty of such an electron satisfies

$$\Delta p \geq \frac{\hbar}{2\Delta x} \approx 5.25 \times 10^{-20}\,\mathrm{kg\,m\,s^{-1}}\ .$$

The confined electron is not going anywhere on average, so its average momentum $\langle p \rangle$ must be zero. Hence

$$(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle = \langle p^2 \rangle\ .$$

Since $c\Delta p = 1.575 \times 10^{-11}\,\mathrm{J} \approx 98\,\mathrm{MeV} \gg mc^2 = 511\,\mathrm{keV}$, the kinetic energy of confinement is sufficient to make the electron highly relativistic. The kinetic energy of the electron is therefore given by the relativisitic formula:

$$KE \;=\; \sqrt{p^2c^2 + m^2c^4} - mc^2 \;\approx\; |p|c\ .$$

To obtain an order of magnitude estimate of this quantity, we approximate $|p|$ by $\Delta p = \sqrt{\langle p^2 \rangle}$ to obtain

$$KE \;\sim\; 98\,\mathrm{MeV}.$$

This argument shows that the KE of an electron confined within a nucleus is of order 100 MeV. Since the kinetic energy of the electron emitted is between 1 and 10 MeV, it seems more likely that the electron was created during the radioactive decay process than that it was confined within the nucleus all along.