Quantum Physics Answer Sheet 4

1. The Rydberg formula is

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{n_\mathrm{f}^2} - \frac{1}{n_\mathrm{i}^2}\right) ,$$

where $\mathrm{R_H} = 1.097 \times 10^7\,\mathrm{m}^{-1}$. In this case, $n_\mathrm{f} = 2$ and $\lambda = c/\nu = (3.00 \times 10^8)/(7.316 \times 10^{14}) = 4.10 \times 10^{-7}\,\mathrm{m}$.

Rearranging the Rydberg formula gives:

$$n_\mathrm{i}^2 = \left(\frac{1}{n_\mathrm{f}^2} - \frac{1}{\mathrm{R_H}\lambda}\right)^{-1} \approx \left(\frac{1}{4} - \frac{1}{1.097 \times 10^7 \times 4.10 \times 10^{-7}}\right)^{-1} \approx 36.1 .$$

Hence, the initial energy level was the $n = 6$ level.

2. The Bohr orbit of the electron must still contain a whole number of de Broglie wavelengths. The angular momentum $L = mvr$ must therefore be quantised just as in a hydrogen atom:

$$L = n\hbar , \qquad n = 1, 2, 3, \dots .$$

However, since the Coulomb attraction between the orbiting electron and the nucleus is larger by a factor Z than in a hydrogen atom, the equation linking centripetal force and centripetal acceleration becomes:

$$\frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r} = \frac{(mvr)^2}{mr^3} .$$

Replacing mvr by $n\hbar$ and rearranging gives the following formula,

$$r_n = \frac{4\pi\epsilon_0 (n\hbar)^2}{Zme^2} ,$$

for the radius of the n^th Bohr orbit.

The total energy of this orbit is

$$\begin{aligned}
E_n &= \mathrm{KE}_n + \mathrm{PE}_n \\
&= \frac{1}{2}mv_n^2 - \frac{Ze^2}{4\pi\epsilon_0 r_n} \\
&= \frac{1}{2}\frac{Ze^2}{4\pi\epsilon_0 r_n} - \frac{Ze^2}{4\pi\epsilon_0 r_n} && \left(\text{since } \frac{mv_n^2}{r_n} = \frac{Ze^2}{4\pi\epsilon_0 r_n^2}\right) \\
&= -\frac{Z^2 me^4}{2(4\pi\epsilon_0\hbar)^2 n^2} && (\text{substituting for } r_n) \\
&\approx -\frac{Z^2 \times 13.6\,\mathrm{eV}}{n^2} .
\end{aligned}$$

3. (i) The shortest wavelength (highest energy) photon that a hydrogen atom can emit ending up in the ground-state ($n_\mathrm{f} = 1$) energy level is produced when the atom decays from an initial state with very large n_i to the $n_\mathrm{f} = 1$ final state. The wavelength of the photon emitted in this transition satisfies the equation

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{1} - \frac{1}{n_\mathrm{i}^2}\right) \approx \mathrm{R_H}\,,$$

and hence

$$\lambda \approx \frac{1}{\mathrm{R_H}} \approx 9.12 \times 10^{-8}\,\mathrm{m}\,.$$

(ii) The shortest wavelength (highest energy) photon that a hydrogen atom can emit without ending up in the ground-state ($n_\mathrm{f} = 1$) energy level is produced when the atom decays from an initial state with very large n_i to the $n_\mathrm{f} = 2$ final state. The wavelength of the photon emitted in this transition satisfies the equation

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{4} - \frac{1}{n_\mathrm{i}^2}\right) \approx \frac{\mathrm{R_H}}{4}\,,$$

and hence

$$\lambda \approx \frac{4}{\mathrm{R_H}} \approx 3.65 \times 10^{-7}\,\mathrm{m}\,.$$

(iii) For the H atom, the normal Rydberg formula applies:

$$\frac{1}{\lambda} = \mathrm{R_H}\left(\frac{1}{n_\mathrm{f}^2} - \frac{1}{n_\mathrm{i}^2}\right)\,,$$

with $\lambda = 121.5\,\mathrm{nm}$. Hence

$$\frac{1}{n_\mathrm{f}^2} - \frac{1}{n_\mathrm{i}^2} = \frac{1}{\lambda \mathrm{R_H}} \approx \frac{1}{121.5 \times 10^{-9} \times 1.097 \times 10^7} \approx 0.75\,.$$

The only solution of this equation with n_i and n_f integers is $n_\mathrm{i} = 2$ and $n_\mathrm{f} = 1$. In other words, the transition is from the first excited state to the ground state.

For the He^+ ion, the energies of the states are $Z^2 = 2^2 = 4$ times what they were in the H atom (see Q2). Hence, the Rydberg formula becomes

$$\frac{1}{\lambda} = 4\mathrm{R_H}\left(\frac{1}{n_\mathrm{f}^2} - \frac{1}{n_\mathrm{i}^2}\right)\,,$$

and the equation for n_f and n_i is

$$\frac{1}{n_\mathrm{f}^2} - \frac{1}{n_\mathrm{i}^2} \approx \frac{0.75}{4} = \frac{3}{16}\,.$$

This equation can be solved by doubling the values of n_i and n_f obtained for the H atom. In other words, the He^+ transition is from the $n_\mathrm{i} = 4$ level to the $n_\mathrm{f} = 2$ level.

4. (i) The potential in this example is peculiar, but a Bohr orbit of radius r still has length $2\pi r$, and the de Broglie wavelength of the particle must still "fit in" to this length:

$$2\pi r = n\lambda \qquad (n \text{ any integer} > 0) .$$

Since $p = mv = h/\lambda$, this condition translates to

$$mvr = mv\frac{n\lambda}{2\pi} = p\frac{n\lambda}{2\pi} = \frac{nh}{2\pi} = n\hbar .$$

In other words, the angular momentum must still be a multiple of $\hbar$.

The next step in deriving the Bohr model of the hydrogen atom is to write down Newton's second law: force = mass $\times$ centripetal acceleration. In this case, the force is

$$F = -\frac{dV}{dr} = -C ,$$

where the $-$ve sign shows that the force acts towards the origin (in the $-r$ direction). Hence, Newton's second law reads:

$$C = \frac{mv^2}{r} = \frac{(mvr)^2}{mr^3} .$$

Using the angular momentum quantisation conditions, $mvr = n\hbar$, then gives

$$C = \frac{\hbar^2 n^2}{mr_n^3} ,$$

or

$$r_n = \left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} ,$$

as required.

(ii) The energy E_n of the n^{th} orbit is:

$$\begin{aligned} E_n &= \text{KE} + \text{PE} = \frac{1}{2}mv_n^2 + Cr_n \\ &= \frac{(mv_n r_n)^2}{2mr_n^2} + Cr_n = \frac{\hbar^2 n^2}{2mr_n^2} + Cr_n \\ &= \frac{\hbar^2 n^2}{2m\left(\frac{\hbar^2 n^2}{mC}\right)^{2/3}} + C\left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} \\ &= \frac{C}{2}\left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} + C\left(\frac{\hbar^2 n^2}{mC}\right)^{1/3} \\ &= \frac{3}{2}\left(\frac{C^2\hbar^2 n^2}{m}\right)^{1/3} , \end{aligned}$$

as required.

Assessed Problem

5. According to the Bohr model, the binding energy and radius of an H atom (treating the nucleus as infinitely massive) in its ground state are

$$E_{\text{binding}} = \frac{me^4}{2(4\pi\epsilon_0\hbar)^2} \approx 13.6\,\text{eV} \,,$$
$$r = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 0.53\,\text{Å} \,.$$

(i) In the case of finite mass nucleus the electron mass m has to be replaced by the reduced mass $m_{\text{reduced}} = (1/m_1 + 1/m_2)^{-1} = m_1 m_2/(m_1 + m_2)$. Hence

$$E_{\text{binding}} = \frac{\frac{m_1 m_2}{(m_1+m_2)} e^4}{2(4\pi\epsilon_0\hbar)^2} \,.$$

so the correct choice is (b).

(ii) and (iii) for the case of positronium the energy and radius of the ground state are given by:

$$E_{\text{binding}} = \frac{0.5me^4}{2(4\pi\epsilon_0\hbar)^2} \approx 6.8\,\text{eV} \,,$$
$$r = \frac{4\pi\epsilon_0\hbar^2}{0.5me^2} \approx 1.06\,\text{Å} \,.$$

(iv) for the case of muonium the mass of the muon is $207m_e$ so the correct reduced mass is $m_\mu m_e/(m_\mu + m_e)$ i.e. can't justify the infinite mass nucleus so well as for the H atom as a muon is around 1/9 of a proton mass. In fact to be strictly correct we need to account for the finite mass of the nucleus for H so then the reduced mass should enter the formula for energy: $m_e m_p/(m_e + m_p)$. In that case the ratio of the ground state energies of muonium compared to H is, going to be the ratio of the two reduced masses: $\frac{m_\mu m_e/(m_\mu+m_e)}{m_e m_p/(m_e+m_p)}$, but to the accuracy needed here the reduced mass for H is just m_e. So the ratio of energies is $207/(1 + 207) = 0.995$.

Tutorial Problem

6. (i) Starting from the definition of the rms momentum,

$$(\Delta p)^2 = \langle (p - \langle p\rangle)^2 \rangle \,,$$

and using the result $\langle p\rangle = 0$ given in the question, we obtain $(\Delta p)^2 = \langle p^2\rangle$. Similarly, we can show that $(\Delta x)^2 = \langle x^2\rangle$. The expression for the total energy,

$$\langle E\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}s\langle x^2\rangle \,,$$

may therefore be rewritten as

$$\langle E \rangle \;=\; \frac{(\Delta p)^2}{2m} + \frac{1}{2}s(\Delta x)^2 \;=\; \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2 \; ,$$

where the last step followed because $\omega = \sqrt{s/m}$.

(ii) The uncertainty principle states that $\Delta x \Delta p \geq \hbar/2$. If we use this to eliminate Δp from the expression for $\langle E \rangle$ we obtain

$$\langle E \rangle \;\geq\; \frac{1}{2m}\frac{\hbar^2}{4(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2 \;=\; \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2 \; .$$

The value of Δx that makes the right-hand side as small as possible (and hence imposes the weakest possible condition on $\langle E \rangle$) may be found by differentiating with respect to Δx and setting the result equal to zero:

$$\frac{d}{d(\Delta x)}\left[\frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2\right] \;=\; \frac{-\hbar^2}{4m(\Delta x)^3} + m\omega^2\Delta x \;=\; 0 \; .$$

Solving this equation gives $\Delta x = \sqrt{\hbar/2m\omega}$. Substituting back into the inequality for $\langle E \rangle$ gives

$$\langle E \rangle \;\geq\; \frac{\hbar^2}{8m(\hbar/2m\omega)} + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right) \;=\; \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega \; ,$$

and hence

$$\langle E \rangle \;\geq\; \frac{1}{2}\hbar\omega \; ,$$

as required.