Quantum Physics Answer Sheet 5

1. The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) \quad = \quad E\psi(x) \ .$$

If the wavefunction

$$\psi_n(x) \quad = \quad \begin{cases} \sqrt{\frac{2}{d}}\sin\left(\frac{n\pi x}{d}\right) & 0 < x < d \\ 0 & \text{otherwise} \end{cases}$$

is a normalised energy eigenfunction, the following conditions hold:

(a) $\psi_n(x)$ satisfies the boundary conditions: $\psi_n(0) = \psi_n(d) = 0$.

(b) $\psi_n(x)$ satisfies the time-independent Schrödinger equation for $0 < x < d$.

(c) $\psi_n(x)$ is normalised.

Consider these conditions one by one:

(a) By inspection, $\psi_n(x)$ satisfies the boundary conditions for $n = 1, 2, \ldots$.

(b) Substitute $\psi_n(x)$ into the left-hand side of the Schrödinger equation for $0 < x < d$:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) \quad &= \quad -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} \qquad [V(x) = 0 \text{ for } 0 < x < d] \\ &= \quad -\frac{\hbar^2}{2m}\sqrt{\frac{2}{d}}\frac{d^2}{dx^2}\sin\left(\frac{n\pi x}{d}\right) \\ &= \quad \frac{\hbar^2}{2m}\left(\frac{n\pi}{d}\right)^2\sqrt{\frac{2}{d}}\sin\left(\frac{n\pi x}{d}\right) \\ &= \quad E_n\psi_n(x) \ , \quad \text{where} \quad E_n \quad = \quad \frac{\hbar^2 n^2\pi^2}{2md^2} \ . \end{aligned}$$

Hence, $\psi_n(x)$ satisfies the time-independent Schrödinger equation in the region $0 < x < d$. The corresponding energy eigenvalue is $E_n = \hbar^2 n^2\pi^2/2md^2$.

(c) If $\psi_n(x)$ is normalised, the integral

$$N \quad = \quad \int_0^d |\psi_n(x)|^2\, dx$$

must be equal to 1. Check this by evaluating the integral:

$$N \quad = \quad \frac{2}{d}\int_0^d \sin^2\left(\frac{n\pi x}{d}\right)\, dx \quad = \quad \frac{2}{d}\times\frac{d}{2} \quad = \quad 1 \ .$$

Hence, $\psi_n(x)$ is normalised.

2. Using the result of Q1, the energy eigenvalues are:

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2md^2} \qquad n = 1, 2, \dots .$$

The energy emitted as the nucleon falls from the $n = 2$ level to the $n = 1$ level is

$$E_2 - E_1 = \frac{3\hbar^2\pi^2}{2md^2} \approx \frac{3 \times (1.05 \times 10^{-34})^2 \times \pi^2}{2 \times 1.67 \times 10^{-27} \times (10^{-15})^2}$$
$$\approx 9.8 \times 10^{-11}\,\mathrm{J} \approx 610\,\mathrm{MeV} .$$

This is a sensible number. The energy released in fission is about 200 MeV per nucleus.

3. (a) The energy difference $\Delta E = hc/\lambda = 6.63 \times 10^{-34} \times 3 \times 10^{-8}/5.5 \times 10^{-7} = 3.62 \times 10^{-19} J$.

(b) We can use again the result derived in Q1 (or the result from lecture which is equivalent):

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2md^2} \qquad n = 1, 2, \dots .$$

The transition is between the n= 2 and n= 1 states so that:

$$\Delta E = \frac{3\hbar^2\pi^2}{2md^2}$$

So we can rearrange this to find d:

$$d = \sqrt{\frac{\hbar^2 n^2 \pi^2}{2m\Delta E}}$$

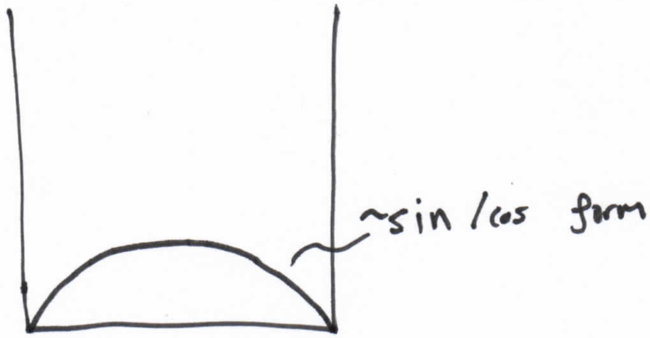
Which gives $d = \frac{3\times(1.05\times^{-34})^2\times\pi^2}{2\times 9.11\times10^{-31}\times3.62\times10^{-19}} = 7.04 \times 10^{-10} m$

(c) $d \propto \sqrt{1/\Delta E} \propto \sqrt{\lambda}$

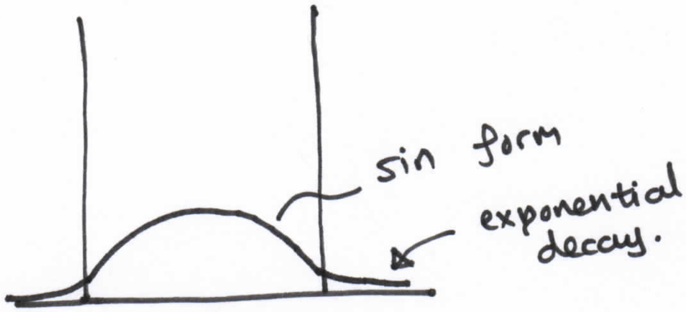
so going to 320 nm emission means that d should reduce by a factor $\sqrt{\frac{320}{550}} = 0.763$, so $d = 5.37 \times 10^{-10} m$.

The depth of well could also be changed as the well is not exactly infinite this will change the energy levels, at the same time the electron effective mass and the levels involved in the transition could also be changed.

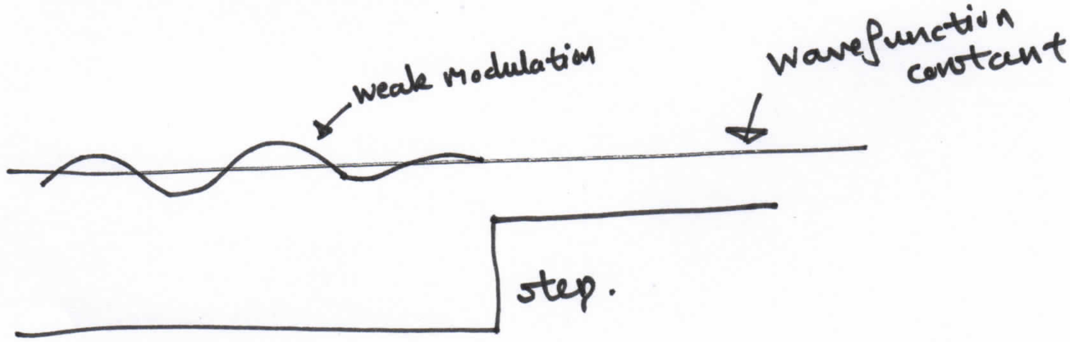
4) (a)


~sin/cos form

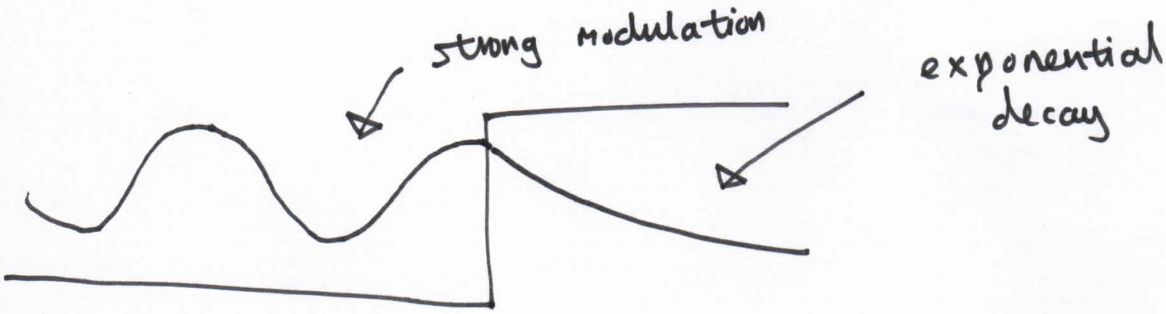
(b)


sin form
exponential decay.

(c)


weak modulation
wavefunction constant
step.

(d)


strong modulation
exponential decay

5) $T = \dfrac{16\, E(V_0 - E)}{V_0^2}\, e^{-2\gamma a}$

where $\gamma = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}$, valid for low current

(i.e. 3×10^{-3} is a low probability).

$E = 50{\cdot}0\,eV$

$V_0 = 70{\cdot}0\,eV$

so the pre-factor is $\dfrac{16 \cdot 50 \cdot 20}{(70)^2} = 3{\cdot}265$

$\Rightarrow \quad e^{-2\gamma a} = 9{\cdot}1875 \times 10^{-4}$

$-2\gamma a = \ln(9{\cdot}1875\times10^{-4}) = -6{\cdot}995$

$a = 6{\cdot}995 / 2\gamma$

where $\gamma = \sqrt{\dfrac{20\times1{\cdot}6\times10^{-19}\times2\times1{\cdot}67\times10^{-27}}{(1{\cdot}05\times10^{-34})^2}} = \sqrt{9{\cdot}694\times10^{23}}$

$= 9{\cdot}846\times10^{11}$

$a = 3{\cdot}55 \times 10^{-12}\,m$

For an electron the pre-factor is the same. We just need to change the mass in the evaluation of γ ($m_e = 9{\cdot}11\times10^{-31}$), so $\gamma = 2{\cdot}30\times10^{10}$

$a = 1{\cdot}52 \times 10^{-10}\,m$

6) $R = \left(\dfrac{k_I - k_{II}}{k_I + k_{II}} \right)^2$ & $T = 1 - R = \dfrac{4 k_I k_{II}}{(k_I + k_{II})}$

where $k_I^2 = \dfrac{2m}{\hbar^2} E$ and $k_{II}^2 = \dfrac{2m}{\hbar^2} (E - V_0)$

(a) $E = 2.1\, V_0$ $\quad (E - V_0 = 1.1 V_0)$

So $R = \left(\dfrac{\sqrt{2.1} - \sqrt{1.1}}{\sqrt{2.1} + \sqrt{1.1}} \right)^2 = \left(\dfrac{0.40}{2.50} \right)^2$ $\quad \sqrt{2.1} = 1.45$, $\sqrt{1.1} = 1.05$

$= 0.026$

(b) $T = 1 - R = 1 - 0.026 = 0.974$

(c) $R = \left(\dfrac{\sqrt{(1.05)} - \sqrt{(0.05)}}{\sqrt{(1.05)} + \sqrt{(0.05)}} \right)^2$ $\quad \sqrt{(1.05)} = 1.025$, $\sqrt{(0.05)} = 0.224$

$= \left(\dfrac{0.801}{1.249} \right)^2 = 0.413$

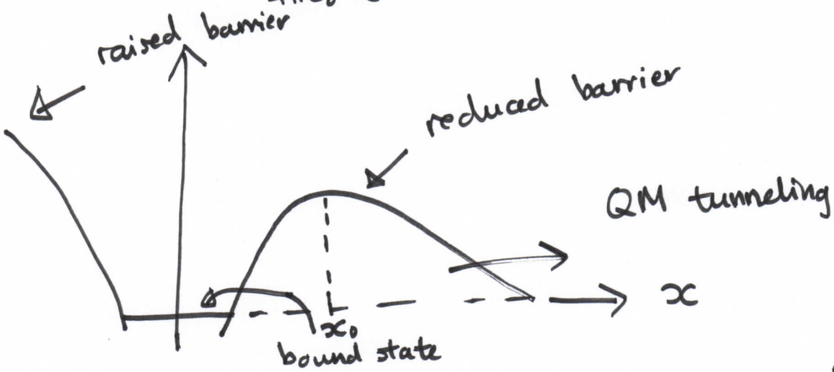
(d) $T = 0.587$

(e) $E < V_0$

effectively $R \to 1$ and $T \to 0$

7) (a) $V = V_c + V_{int}$ $\qquad (E = F_0 \cos \omega t)$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{x} + e x E_0$$



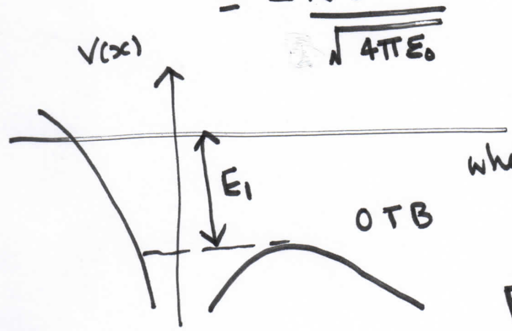
(b) The turning point in the barrier is where: (use $E_0 = F_0$ to avoid confusion with energy)

$$\frac{dV(x)}{dx} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{x^2} - eF_0 = 0$$

So $\quad x^2 = \dfrac{e}{4\pi\varepsilon_0 F_0} \quad \Rightarrow \quad x_0 = \sqrt{\dfrac{e}{4\pi\varepsilon_0 F_0}}$

$$V(x_0) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{x_0} - e x_0 F_0 = -\frac{1}{4\pi\varepsilon_0} e^2 \sqrt{\frac{4\pi\varepsilon_0 F_0}{e}} - e\sqrt{\frac{e}{4\pi\varepsilon_0 F_0}} \cdot F_0$$

$$= -\frac{2 e^{3/2} F_0^{1/2}}{\sqrt{4\pi\varepsilon_0}}$$



when $V(x_0) = E_1 = 13.6\text{eV}$ for H ground state in Bohr atom

$$F_0 = \frac{4\pi\varepsilon_0 E_1^2}{4e^3} = \frac{\pi\varepsilon_0 E_1^2(\text{eV})}{e}$$

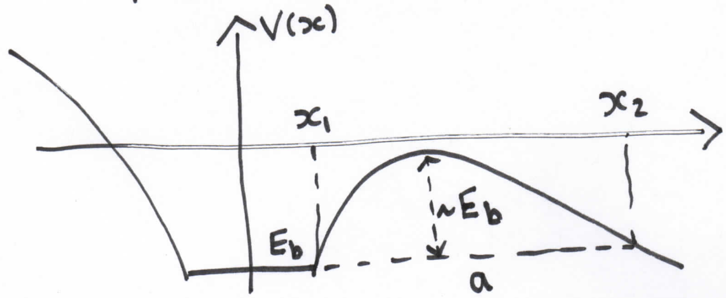
$$= 3.21 \times 10^{10}\,\text{Vm}^{-1}$$

Such fields are routinely reached by focusing a high power laser. It corresponds to a focused intensity $\sim 1.4 \times 10^{14}\,\text{Wcm}^{-2}$ – there are a half dozen lasers in Blackett that can do this!

(c) This is harder, we need a more sophisticated treatment to solve it, but we will try.

Let us make the assumption that we can approximate by using our rectangular barrier formula:

$$T \propto e^{-2\gamma a}$$



For a weak distortion

$$\gamma = \sqrt{\frac{2M E_b}{\hbar^2}}$$

$(V_0 \simeq 0)$

$$V(x) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{x} + e x F_0 = E_b$$

x_1 and x_2 can be found as the solutions of a quadratic equation:

$$\left(x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \right)$$

$$+\frac{e^2}{4\pi\varepsilon_0} + E_b x + e F_0 x^2 = 0$$

Using $$x = \frac{-E_b \pm \sqrt{(E_b^2 - 4e^3F_0/4\pi\varepsilon_0)}}{2eF_0}$$

$$a = x_2 - x_1 = \frac{2\sqrt{(E_b^2 - 4e^3F_0/4\pi\varepsilon_0)}}{2eF_0}$$

For weak fields $E_b^2 >> 4e^3F_0/4\pi\varepsilon_0$

$$a \simeq \frac{E_b}{eF_0}$$

$$T \propto e^{-\sqrt{\frac{2M}{\hbar^2}}\cdot\frac{1}{e}\frac{E_b^{3/2}}{F_0}}$$

this has same form as rigorous theory.