# 16 Quantum Harmonic Oscillator

## 16.1 Classical Harmonic Oscillator

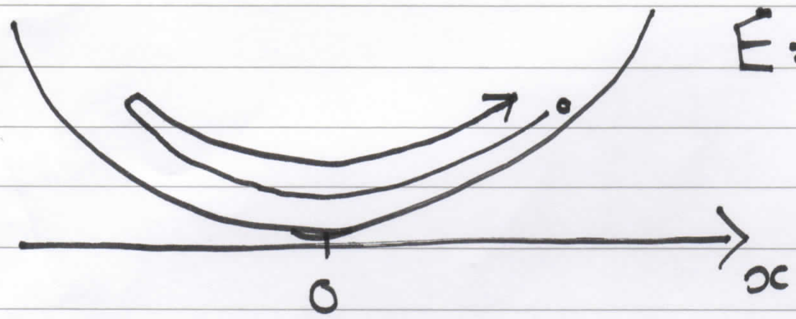
or simple harmonic oscillator SHO

$$F = -kx \quad [16.1]$$ k – 'stiffness' constant

$$V(x) = \tfrac{1}{2}kx^2 \quad [16.2]$$ parabolic in $x$ ($F = -dV/dx$)

$$\omega_0 = \sqrt{\frac{k}{m}} \quad [16.3]$$ natural frequency ($m$ – mass or $\mu$ – reduced mass)

$$\therefore\ k = m\omega_0^2 \quad \text{so} \quad V(x) = \tfrac{1}{2}m\omega_0^2 x^2 \quad [16.4]$$




$$E = \frac{p^2}{2M} + \frac{1}{2}kx^2 \quad [16.5]$$

k.e ⇄ p.e

Classical position probability distribution $P(x)$


$$E = \tfrac{1}{2}kx_{max}^2$$

- Many potential wells are a good approximation to SHO (Taylor expansion around a minimum)
- Mass on spring, pendulum, vibrations in a molecule etc.

MPH

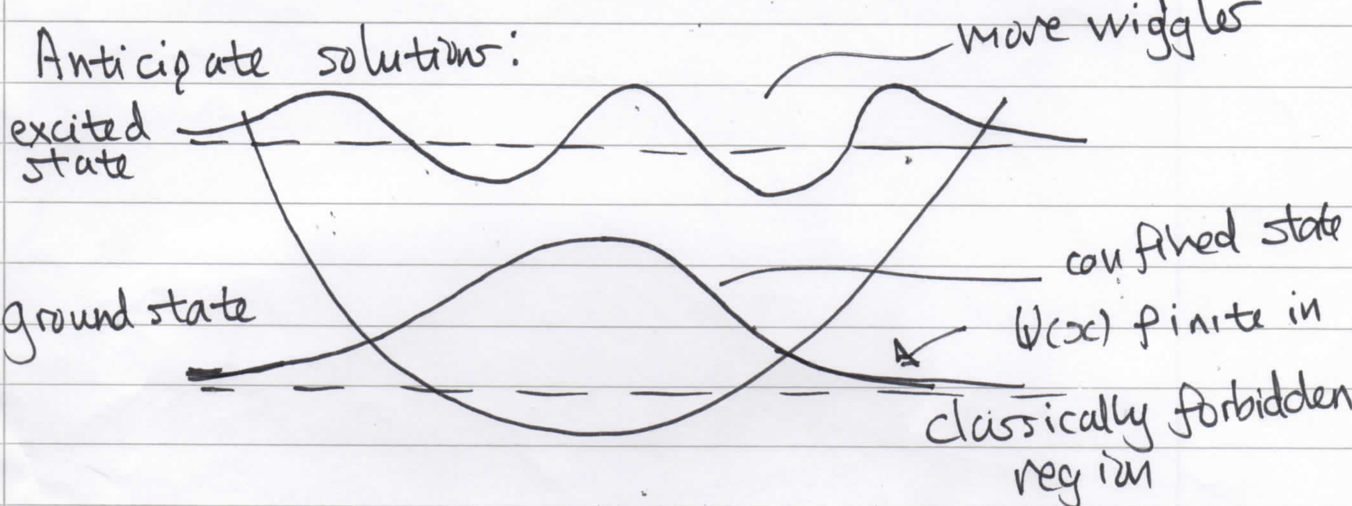
## 16·2 Quantum Harmonic Oscillator QHO

Time-independent S-equan.

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{1}{2} m\omega_0^2 x^2 \psi(x) = E\psi(x)} \quad 16·6$$

[Simple enough → but you need more practice with 2nd order differential equation to solve it]

Anticipate solutions:




Lowest state $\psi_0(x) = C\exp\left(\frac{-x^2}{\Delta x^2}\right)$ [16·7]

Gaussian form

More rigorously:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}x^2\right)} \quad [16·8]$$

controls width of distribution

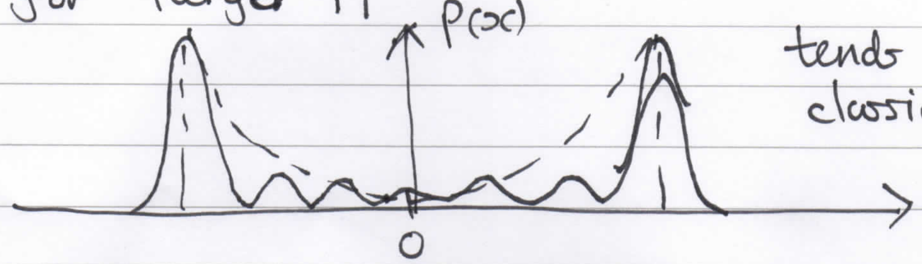
$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{2\hbar}x^2\right)} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) \quad [16·9]$$

Hermite polynomials – give the wiggles

[wait for 2nd Year!]

$$P(x) = |\psi_n(x)|^2$$

$P(x)$ for large $n$




tends to classical

An example of <u>Correspondence Principle</u>

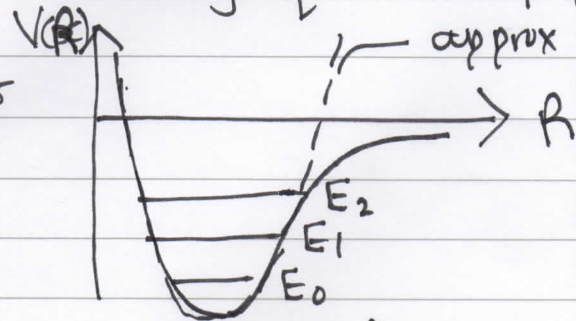
## 16.3 Energy Eigenvalues of QHO

$$\boxed{E_n = (n + \tfrac{1}{2})\hbar\omega_0} \qquad [16.10] \quad n = 0, 1, \ldots$$

- confined system so (not surprisingly) energy quantised
- <u>evenly spaced energies</u> $\Delta E_{n,n+1} = \hbar\omega_0$ [16.11]
- $\boxed{E_0 = \tfrac{1}{2}\hbar\omega_0}$ <u>Zero point energy</u> [16.12]

  inherently quantum property

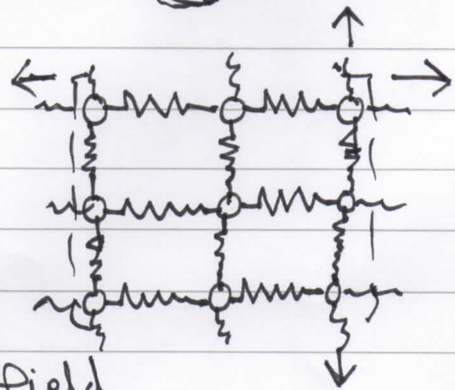
<u>Examples</u> – Molecules




<u>vibrational energies</u>

$E_n \simeq (n + \tfrac{1}{2})\hbar\omega_0$

– Lattice vibration



<u>phonons</u> obey same rules

– modes of radiation field

<u>photons</u> $E_n = (n + \tfrac{1}{2})\hbar\omega_0$

└ degree of mode excitation

1 photon $E = \hbar\omega_0$

(slides)