17 Energy and Time in QP

17.1 Energy-Time Uncertainty Relation

From free particle wave:

$$\psi(x,t) = A \exp\left[i\left(\frac{p}{\hbar}x - \frac{E}{\hbar}t + \phi\right)\right] \quad [17.1]$$

We see phase evolves as $\left(\frac{E}{\hbar}t\right)$ or ωt

(L3) From E·M theory we have <u>bandwidth theorem</u> for t & ω: (in terms of variance rather than FWHM)

$$\Delta t \, \Delta\omega \gtrsim \tfrac{1}{2} \quad [17.2]$$

Leads to an uncertainty relation for <u>energy-time</u>

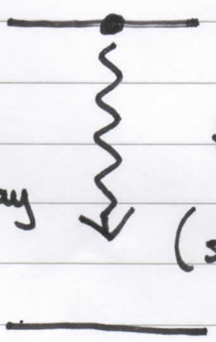
$$\Delta t(h\Delta\nu) = \Delta t(\hbar\Delta\omega) = \boxed{\Delta t\,\Delta E \gtrsim \frac{\hbar}{2}} \quad [17.3]$$

(note no operator for t in QM) (note sometimes: $\Delta t \Delta E \gtrsim \hbar$)

Examples:-

Life-times of states that decay can be accessed by <u>spectral</u> (energy domain) measurements. (e.g. radioactive decay, fast electronic state decay in matter) spontaneous emission)

excited state



U_b

eg H atom τ_{decay}

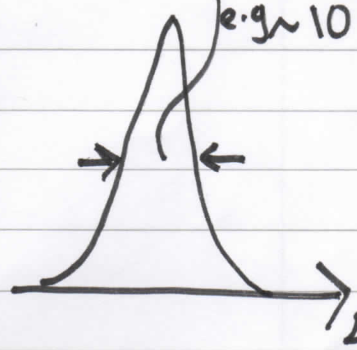
<u>spontaneous</u> <u>emission</u>

(stimulated by vacuum field)

U_a

$(e^{-t/\tau})$

γ spectral width e.g. $\sim 10^8$ Hz



ν

can do a bit of maths to link τ + γ rigorously - we forgoe this...

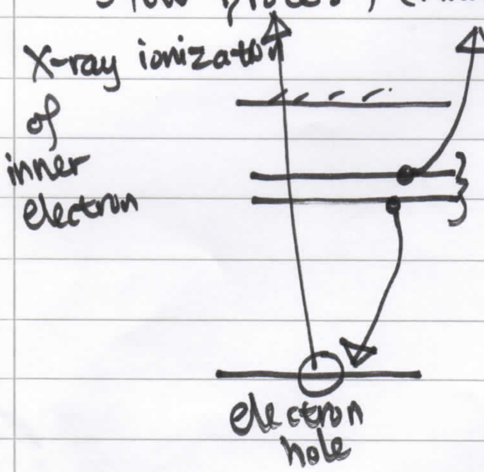
Typical $\quad \Delta\nu \simeq 5\times10^{7}\ \text{Hz}$

$$\Delta E = 3\cdot315\times10^{-26}\,\text{J}$$

$$\tau \simeq \Delta t \simeq \frac{10^{-34}\ (\text{Js})}{2\times3\cdot315\times10^{-26}\ (\text{J})} = 1\cdot51\times10^{-9}\,\text{s} \quad (1\cdot5\,\text{ns})$$

(c.f. $\sim 10^{-15}$ s cycle time)

(spontaneous emission of photon is a (relatively) slow process) (missed in QM without Light Quantization – but not a bad approx).



Auger decay – due to electron-electron interaction

$\Delta E \simeq 0\cdot5\,\text{eV}$

$$\Delta t \gtrsim \frac{\hbar}{2\times0\cdot5\times1\cdot6\times10^{-19}} \simeq 6\cdot25\times10^{-16}\,\text{s}$$

fast process $\rightarrow$ $\lesssim 1\,\text{fs}$

17.2 Time-Dependent Schrödinger Equation

(from L12)

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) \qquad [17\cdot4]$$

$\hat{H}$ time-independent

If $\Psi(x,t)$ is an eigenstate of Hamiltonian $\hat{H}$ we can write:

$$\hat{H}\Psi_n(x,t) = E_n\Psi_n(x,t) \qquad [17\cdot5]$$

E_n – nth eigenvalue, Ψ_n – nth eigenstate

So
$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi_n(x,t) = E_n\Psi_n(x,t)} \qquad [17\cdot6]$$

We can factor $\Psi_n(x,t)$ into a spatial part $u_n(x)$ and a temporal part $f(t)$

that satisfies 17·6 i.e $f(t) = e^{-\frac{iE_n t}{\hbar}}$

$$i\hbar \frac{\partial}{\partial t}\left(u_n(x)\, e^{-\frac{iE_n t}{\hbar}} \right) = u_n(x)\, i\hbar \frac{\partial}{\partial t} e^{-\frac{iE_n t}{\hbar}}$$

$$= u_n(x).\ i\hbar . -\frac{i}{\hbar} E_n\, e^{-\frac{iE_n t}{\hbar}}$$

$$= E_n\, u_n(x)\, e^{-\frac{iE_n t}{\hbar}}.$$

- Temporal evolution of an energy eigenstate always of form $e^{-\frac{iE_n t}{\hbar}}$
- $\frac{E_n t}{\hbar}$ is the phase accumulated, phase evolution $\phi(t)$ faster or slower depending on E_n.
- $P_n(x,t) = |\psi_n(x,t)|^2 = |u_n(x)|^2 e^{-\frac{iE_n t}{\hbar}} . e^{+\frac{iE_n t}{\hbar}}$

$$= |u_n(x)|^2$$

a <u>stationary state</u>, doesn't change with time.

In general ~~[illegible]~~ there ~~[illegible]~~ is evolution in time of the properties of a quantum state:

— If $\psi(x,t)$ is a superposition of eigenstates

— If there is a time dependent Hamiltonian

$\hat{H}(x,t)$

e.g. a time-dependent E·M field.