First Year Special Relativity - Lecture 4
The Lorentz transformations

Paul Dauncey, 15 May 2018

1 Introduction

The material in this lecture is covered in Young and Freedman, Sec. 37.5 and in McCall in Sec. 5.11 and 5.12.

We have seen that the postulates of relativity lead to several unexpected consequences, like time slowing down, objects contracting and things not being simultaneous for different observers. We now want to look at the general equations which transform points in space and time as all the other effects actually result from the application of these general transformations. These equations are called the 'Lorentz transformations'.

2 Events

When we looked at rotations, the equations change a point at x, y to x', y', i.e. they move a specific point in space. We will see that the Lorentz transformations are mathematically similar and they must also work on a particular point. However, this is not a point just in space but in space and time. Like rotations, Lorentz transformations can also be active or passive, where active would require physically changing the speed of an object (in the jargon, doing a 'boost'), while passive is considering how the object would appear to observers moving at different speeds. If just considering observers moving along the x axis, then we actually only have to consider values of t and x. Hence, we need to consider something which happened at a specific x at a particular time t. Such an occurrance is called an 'event'. This is a fundamental concept in relativity and as long as you can break anything down into a list of events, you can usually solve any relativity problem. Note, a true event occurs (in principle) at an infinitessimallly small point in space at an instant in time. Obviously, as long as events are small and short compared with the scales of the system being considered, then they are a good approximation to the ideal.

In terms of events, then the 'all-seeing' observer is required to be able to measure the t and x of any event which occurs. Also, we can now make more precise statements about proper time and length. Proper time is the time between two events in an inertial frame in which the events have the same position. Similarly, proper length is the distance between two events in the frame in which the events have the same time and in which the object they mark the ends of is stationary.

3 The Lorentz transformations

Given an event at t, x, y, z, then a passive Lorentz transformation from an initial inertial frame to the frame of an observer moving at speed v along the $+x$ axis can be written as

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \qquad x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$ as defined previously. The values of t' and x' tell you the coordinates of the event in the different frame. One critical thing to note is that the change from t to t' does not only depend on the speed (through v and so also γ) but also on the position x. This is the basic cause of many of the tricky effects in relativity, such as non-simultaneity.

Since y and z are unchanged by an observer moving along x, we will usually not bother to write them except when required. Note, in a similar way to rotating around the origin, for simplicity we have chosen the origin so that $t = 0, x = 0$ always transforms to $t' = 0, x' = 0$. Using the definition introduced previously of $\beta = v/c$, so $\gamma = 1/\sqrt{1-\beta^2}$, the transformations can also be written as

$$t' = \gamma\left(t - \frac{\beta x}{c}\right), \qquad x' = \gamma(x - \beta ct)$$

It is obvious that t and x have different dimensions and this makes the equations a little less intuitive. Obviously, as distance is speed times time, then ct has the same dimensions as x so we can rewrite the above by multiplying the first equation throughout by c

$$ct' = \gamma(ct - \beta x), \qquad x' = \gamma(x - \beta ct)$$

which then shows the symmetry between ct and x quite clearly. We can also write these two equations as one matrix equation

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

so that the similarity to a rotation is more obvious. Clearly this is *not* actually a rotation though. All the elements of a rotation matrix are sines and cosines and so must have a magnitude of one or less. However, we know γ has a minimum (not maximum) of one and can be arbitrarily large. This means $\gamma\beta$ can also be very large too. Note also that the off-diagonal terms in a rotation matrix have an equal magnitude but opposite sign, whereas here they have the same sign. Figure 1 compares the way points under rotations and Lorentz transformation change.

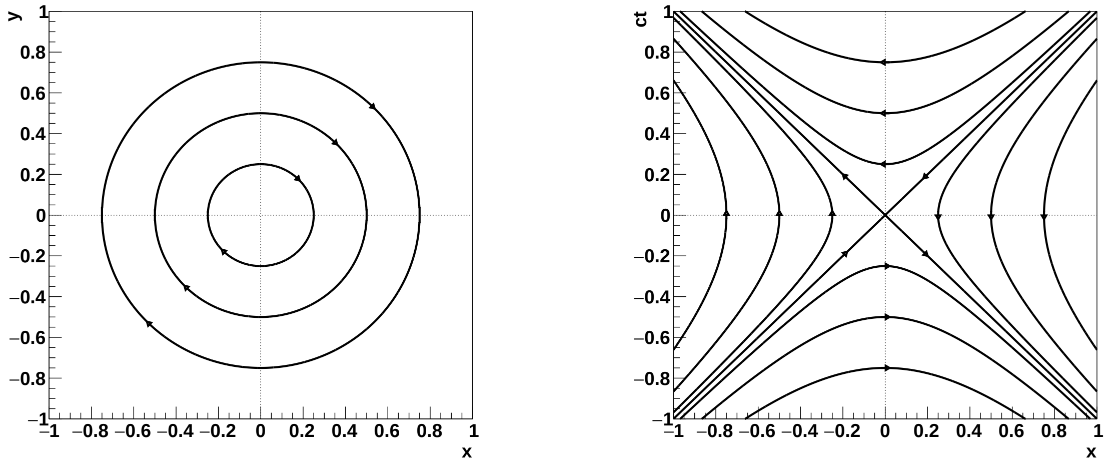

Figure 1: Left: Change of x, y coordinates under rotations. Right: Change of ct, x coordinates under Lorentz transformations. In both cases, a point on one of the lines will move along that line under the relevant transformation.

How would we do the inverse transform to change back to the original observer? As for the rotation case we can invert the matrix. Using the usual 2×2 method, we first find the determinant

$$\Delta = (\gamma)(\gamma) - (\gamma\beta)(\gamma\beta) = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1$$

where the last step is simply due to the definition of γ. Hence

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

Since $\gamma(-\beta) = 1/\sqrt{1-(-\beta)^2} = 1/\sqrt{1-\beta^2} = \gamma(\beta)$ then, just as for rotations, the inverse is simply a transformation with the same magnitude but opposite sign, i.e. $\beta \to -\beta$, which is what intuition would tell us.

There is another way that the Lorentz transformation is commonly written, using a variable called 'rapidity' η. This is related to speed but is not identical. In terms of η, the Lorentz transformations can be written to look quite similar to rotations using hyperbolic functions

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

This works because $\gamma^2 - \gamma^2\beta^2 = 1$ requires $\cosh^2\eta - \sinh^2\eta = 1$, which does indeed hold for the hyperbolic functions. These equations in fact actually define η, meaning for a given β, then η is defined by

$$\tanh\eta = \frac{\sinh\eta}{\cosh\eta} = \frac{\gamma\beta}{\gamma} = \beta$$

4 Small speed approximation

We also need to be sure an Lorentz transformation corresponds to the classical Galilean transforms for small boosts. If β is small enough, then $\gamma \approx 1$. Hence, an Lorentz transformation is then approximately

$$ct' \approx ct - \beta x, \qquad x' \approx x - \beta ct$$

Since $\beta = v/c$, the second equation can be written as

$$x' \approx x - vt$$

which is what we got for x' classically. The first equation can be written as

$$t' \approx t - \frac{v}{c}\frac{x}{c}$$

and so has an extra term compared with the classical expression $t' = t$. However, for the small speeds we are used to, $v \ll c$, and for our normal day-to-day experiences, any distance x gives an extremely small time value when divided by c. Hence, this extra term consists of the product of two very small numbers and so $t' \approx t$ to a very good approximation, as assumed in the Galilean transformations.

5 Consistency of the speed of light

I have claimed a Lorentz transformation is the correct relativistic transformation. If this is true then it must obey the second postulate, specifically the speed of light must be the same after the transformation. Say (for convenience) a light pulse is generated at the origin $\vec{r} = 0$ at $t = 0$ and has a velocity $\vec{u}$ where $|\vec{u}|^2 = c^2$. After a time T, it will be at $\vec{r} = \vec{u}T = (u_xT, u_yT, u_zT)$. Hence, take its position at $t = 0$ as one event and at $t = T$ as a second event. Since all the coordinates are zero for the first event, this event in another frame also has all the coordinates at zero. For the second event, then

$$cT' = \gamma(cT - \beta u_xT), \qquad x' = \gamma(u_xT - \beta cT), \qquad y' = u_yT, \qquad z' = u_zT$$

It has taken a time T' to go from the origin to $\vec{r}'$ which is a total distance d given by

$$d^2 = x'^2 + y'^2 + z'^2 = \gamma^2(u_xT - \beta cT)^2 + u_y^2T^2 + u_z^2T^2 = \gamma^2u_x^2T^2 - 2\gamma^2\beta cu_xT^2 + \gamma^2\beta^2c^2T^2 + u_y^2T^2 + u_z^2T^2$$

But $u_x^2 + u_y^2 + u_z^2 = c^2$ so

$$d^2 = (\gamma^2 - 1)u_x^2 T^2 - 2\gamma^2 \beta c u_x T^2 + \gamma^2 \beta^2 c^2 T^2 + c^2 T^2 = (\gamma^2 - 1)u_x^2 T^2 - 2\gamma^2 \beta c u_x T^2 + (\gamma^2 \beta^2 + 1)c^2 T^2$$

Noting again that $\gamma^2 - \gamma^2\beta^2 = 1$, then $\gamma^2 - 1 = \gamma^2\beta^2$ and $\gamma^2\beta^2 + 1 = \gamma^2$. Hence

$$d^2 = \gamma^2 \beta^2 u_x^2 T^2 - 2\gamma^2 \beta c u_x T^2 + \gamma^2 c^2 T^2 = \gamma^2 (cT - \beta u_x T)^2 = c^2 T'^2$$

Therefore, in the boosted frame, the two events are precisely the right distance apart for the light pulse to travel between them in time T'. Hence an Lorentz transformation preserves the speed of light and so does not violate the second postulate.

6 Velocity transformation

It is important to understand that an Lorentz transformation by $\beta = v/c$ does *not* mean all velocities change from u to $u' = u - v$ as in the Galilean transformations. An explicit case is the speed of the light pulse, found above, which does not change speed at all.

Let's do the similar calculation to the light speed one above, but for an object moving at a speed $u < c$ along the $+x$ axis, to keep it simple. In time T it goes from the origin to uT. Hence, applying an Lorentz transformation to this second event gives

$$cT' = \gamma(cT - \beta uT), \qquad x' = \gamma(uT - \beta cT)$$

The speed in the boosted frame is $u' = x'/T'$ which is

$$u' = \frac{x'}{T'} = \frac{\gamma(uT - \beta cT)}{\gamma(cT - \beta uT)/c} = \frac{u - \beta c}{1 - \beta u/c} = \frac{u - v}{1 - uv/c^2}$$

This is the velocity transformation formula. Note, although calculated from the Lorentz transformations, it is *not* itself a Lorentz transformation; velocity does not change in the same way as position. Another way it is often written is in terms of $\beta_u = u/c$, for which

$$\beta'_u = \frac{\beta_u - \beta}{1 - \beta_u \beta}$$

To avoid any confusion: β_u is the object speed and β the transformation parameter.

A few other points should be noted:

1. If the initial velocity is $u = 0$, then $u' = (-v)/1 = -v$ i.e. boosting to a frame going at v along the $+x$ axis makes an object initially at rest appear to have $-v$, as we would expect. Similarly, a boost by $-v$ makes an object initially at rest appear to have velocity $+v$.

2. If the original velocity was in fact the speed of light, i.e. $u = c$, then $u' = (c - v)/(1 - cv/c^2) = c(1 - v/c)/(1 - v/c) = c$ and so is unchanged, as we found before.

3. For a small velocity $u \ll c$ and small boost $v \ll c$, then uv/c^2 is extremely small and hence $u' \approx u - v$, which again agrees with the Galilean result.

4. Doing a second Lorentz transform will allow us to calculate u'' in terms of u' and hence in terms of u using the above formula. Trying this will show that

   $$u'' \neq \frac{u - (v_1 + v_2)}{1 - u(v_1 + v_2)/c^2}$$

   but gives a more complicated expression. Hence, unlike in classical physics, velocities in subsequent Lorentz transforms do not simply add.