First Year Special Relativity - Lecture 10
The relativistic Doppler effect

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1 Introduction

The material in this lecture is covered in Young and Freedman Sec. 37.6 and in McCall in Sec. 6.7.

You will know about the Doppler effect in sound; it is pretty obvious whenever an ambulance passes you in the street. Waves emitted from an object approaching you get bunched up and so have a shorter wavelength, giving a higher frequency, while the opposite is true as the object moves away from you. The same effect also applies to light. However, for objects moving at relativistic speeds, there is an additional effect due to time dilation which needs to be taken into account first.

2 Time dilation

Consider a relativistic object emitting light where the light is travelling purely perpendicular to the direction of motion of the object. Classically there would be no Doppler shift. This is equivalent to the moment the ambulance is closest to you; at that time you hear the "true" frequency.

However, we know that time for a moving object is dilated, and so is slowed relative to the observer. Hence, if the object emitted light of a frequency f in its rest frame, it will actually emit light of frequency $f_d = f/\gamma_u$ when moving perpendicular to an observer and hence the observer will always measure a lower frequency in this orientation. It terms of colours, this light is moved from the blue end of the spectrum towards the red end, and this is called 'red-shifted' light, as opposed to 'blue-shifted' which is when light is pushed higher in frequency.

3 The stretch or squeeze of wavelengths

We now need to consider the effect of the wavelength of light being squeezed or stretched. This is effectively just geometry and so also holds relativistically. The frequency of light observed depends on whether the observer is in front (or forward) of the source in its direction of motion, or behind (or backward), as shown in Fig. 1

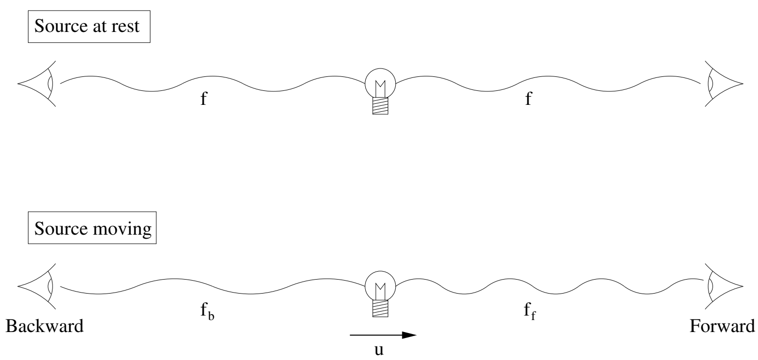

Figure 1: A moving source gives different frequencies to a forward (f) and backward (b) observer.

Specifically, assume the light source is moving with speed u along the x axis. Consider one full wavelength of light also emitted in the $+x$ direction as shown in Fig. 2. Emitting one wavelength will take a time equal to one period $T = 1/f_d$; note we must use f_d as this is the source frequency in the observer frame.

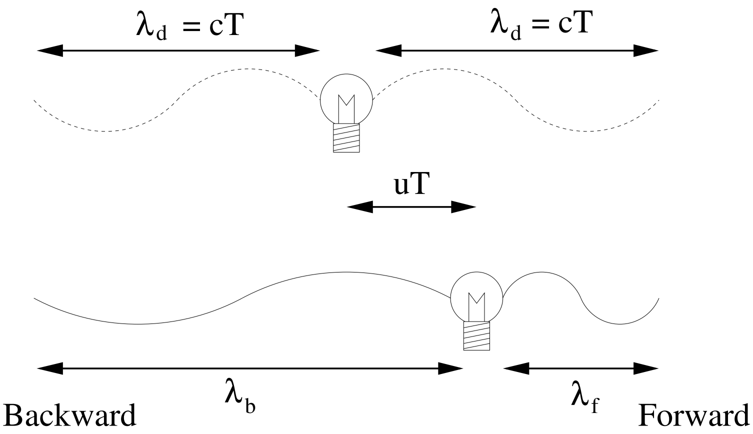

Figure 2: Emission of one wavelength of light for a source moving with speed u. Top: The dashed lines show the wave which would have been emitted if the source had not been moving to the right. Bottom: The actual waves emitted by the moving source.

During this time, the leading part of this wave will have gone a distance cT while the emitting object will have gone uT. Hence the wave for one period will be squashed into a length $(c-u)T$, i.e. this is its wavelength. Hence

$$\lambda_f = (c-u)T = \frac{c-u}{f_d} = \frac{c\gamma_u(1-\beta_u)}{f} = \gamma_u(1-\beta_u)\lambda$$

where the subscript f indicates this is as seen by an observer in front of the moving source. Its frequency will therefore be

$$f_f = \frac{c}{\lambda_f} = \frac{c}{\gamma_u(1-\beta_u)\lambda} = \frac{f}{\gamma_u(1-\beta_u)}$$

If the light is going in the $-x$ direction, then the wave from one period will fit into a length $(c+u)T$ so

$$\lambda_b = \gamma_u(1+\beta_u)\lambda \qquad \text{and} \qquad f_b = \frac{f}{\gamma_u(1+\beta_u)}$$

where b indicates backwards, i.e. the observer is behind the moving source. Note, these are exactly the same equations with β_u replaced by $-\beta_u$, as might be expected.

There is another common formula for the above, using the explicit form for γ_u. Consider

$$\gamma_u(1-\beta_u) = \frac{1-\beta_u}{\sqrt{1-\beta_u^2}} = \sqrt{\frac{(1-\beta_u)^2}{(1-\beta_u)(1+\beta_u)}} = \sqrt{\frac{1-\beta_u}{1+\beta_u}}$$

Similarly

$$\gamma_u(1+\beta_u) = \frac{1+\beta_u}{\sqrt{1-\beta_u^2}} = \sqrt{\frac{(1+\beta_u)^2}{(1-\beta_u)(1+\beta_u)}} = \sqrt{\frac{1+\beta_u}{1-\beta_u}}$$

This tells us that

$$\gamma_u(1-\beta_u) = \frac{1}{\gamma_u(1+\beta_u)}$$

Note that these two factors, which are the inverse of each other, are also simply related by $\beta_u \to -\beta_u$. Hence, we can write

$$\lambda_f = \lambda\sqrt{\frac{1-\beta_u}{1+\beta_u}} \qquad \text{and} \qquad f_f = f\sqrt{\frac{1+\beta_u}{1-\beta_u}}$$

while

$$\lambda_b = \lambda\sqrt{\frac{1+\beta_u}{1-\beta_u}} \qquad \text{and} \qquad f_b = f\sqrt{\frac{1-\beta_u}{1+\beta_u}}$$

In this form, it is straightforward to see that, assuming β_u is positive, then $f_f > f$, i.e. this is always a blue-shift, while $f_b < f$, i.e. this is always a red-shift.

Hence, it is very important to be sure of the sign of β_u and what it means, as it is easy to misinterpret a Doppler equation. It is always best to think about whether physically the light wavelength will be squeezed (blue-shifted), when the source moves towards the observer, or stretched (red-shifted), when it is moving away. This will normally ensure that you get the right sign for β_u.

We know the first postulate of relativity states all inertial frames are equivalent. Hence, the above, which relate to a moving source and stationary observer, must also hold (from the observer's perspective) in another inertial frame where the source is stationary and the observer is moving towards or away from the source. Note that this is very different from sound. In that case, you have to specify whether the source or observer (or both) are moving because the sound moves at a fixed speed relative to the air. In this respect, the relativistic Doppler shift is simpler as light always goes at speed c and all inertial frames are equivalent so it doesn't matter whether the source or the observer are moving; in fact we cannot even define this because, as always, it is purely a matter of which inertial frame the source and observer are in.

As you will know, the Universe is expanding, which results in galaxies further from us on average appearing to move away faster than ones closer to us. The speeds the galaxies are moving away can be measured by the (usually) red shift of specific lines in spectra from atoms. While part of this effect is related to the Doppler shift discussed above, some of it is also a General Relativity effect of the actual space between us and the other galaxies being stretched by the expansion itself.

4 Doppler shift from four-momentum

We can consider the Doppler effect from a different perspective by considering it in terms of photons. Consider a stationary tank firing a shell which has speed u. Classically, if the tank starts moving with speed v in the direction it is firing, the next shell will have speed $u + v$ and so will have a higher momentum and hence higher kinetic energy. The shell is "thrown forwards" by the tank's motion and photons work in exactly the same way. Although we cannot speed up photons (as they always go at the speed of light), a moving source can still given extra momentum and hence energy to photons emitted in the same direction. Therefore, let's try to find the Doppler shift result a different way using the fact that we know about the four-momentum and quantum mechanics.

Consider a photon emitted in the source rest frame along the $+x$ axis. The light has frequency f and hence the photons in the light have energy $E = hf$ and a momentum $pc = hc/\lambda = hf$ also. If we do a Lorentz transformation, we can move to a frame where the source is moving. As always, when we do a Lorentz transformation with speed v on a object at rest (here the source), the object speed is $u = -v$ in the second frame. Transforming the photon then gives

$$E' = \gamma(E - \beta pc) = \gamma_u(E + \beta_u pc) = \gamma_u(hf + \beta_u hf) = hf\gamma_u(1 + \beta_u)$$

and since $E' = hf_f$

$$f_f = f\gamma_u(1+\beta_u) = \frac{f}{\gamma_u(1-\beta_u)}$$

which is the same result as previously. The above assumed pc was positive; putting it negative would be equivalent to the photon travelling in the other direction and so would give f_b.

Note this is often by far the easiest way to calculate the Doppler shift, particularly for photons emitted at arbitrary angles. However, this method doesn't clearly bring out the two different physical effects which contribute; namely time dilation and the squeeze/stretch of the wavelengths. Importantly, the Lorentz transformation method is also usable for particles with non-zero mass; for example, the Doppler shift for the quantum wavelength of a moving electron can be easily found using four-vectors.

There is one subtlety which might be overlooked in this calculation; it only depends on the photon transformation and actually has nothing to do with the source motion. The calculation would be identical if the source was stationary or moving in the original frame; $-\beta$ is then just the transformation parameter between frames, not the final speed of the source. This is effectively again saying that all frames are equivalent, as discussed above.

We might also want to check the case considered in Sec. 2, i.e. when the motion is transverse to the photon and the only effect is time dilation. For this case, the photon must be emitted at 90° to the source motion but this has to be in the *observer* frame, not the source rest frame. This means that, while the photon must have a non-zero $p'_y = p_y$, we need $p'_x = 0$. This requires the photon to have a specific p_x such that the Lorentz transformation reduces it to zero, and this specific value is given by

$$p'_x c = \gamma(p_x c - \beta E) = \gamma_u(p_x c + \beta_u E) = 0 \qquad \text{so} \qquad p_x c = -\beta_u E$$

Hence

$$E' = \gamma(E - \beta p_x c) = \gamma_u(E + \beta_u p_x c) = \gamma_u(E - \beta_u^2 E) = \frac{E}{\gamma_u}$$

and so, since $E' = hf_d$ in the previous notation, then $f_d = f/\gamma_u$ as stated previously.