Mechanics 2017

Problem sheet 7

24 November 2017

Assessed problem

You need to answer this problem on Blackboard Learn. Type numerical values into the boxes provided. You might need to use the departmental formula sheet.

A satellite is in a circular orbit around the Earth's equator; from the Earth's surface, it appears to be stationary. These are termed "geostationary" orbits.

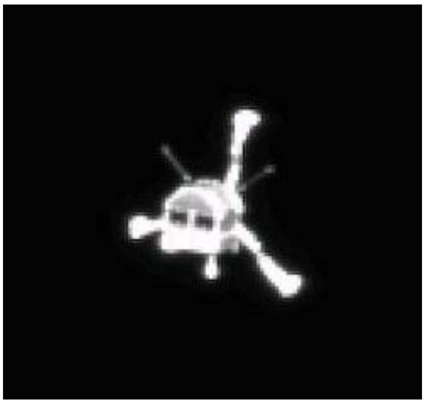
(a) What is the radius of its orbit from the Earth's centre, in km?

(b) Hence calculate at what angle to the vertical a satellite dish should be inclined in London.

(c) If such a satellite is used for phone conversations, what is the minimum expected round-trip delay (times it takes for a signal to travel person A-satellite-person B-satellite-person A), in milliseconds?

(d) What is the equivalent delay for an undersea fibre-optic link between London and Orlando, in milliseconds? The refractive index of the fibre is 1.4, so the signal travels at the $1/1.4$ of the speed of light in vacuum.

In 2014, the Philae lander was released by Rosetta around 22 km from the centre of comet Churyumov-Gerasimenko, whose mass is 9×10^{12}kg, from where it dropped onto the surface around 19 km below.



(e) Assuming Philae started from rest and the comet is a homogeneous spherical mass, how fast was the lander travelling when it reached the surface?

(f) From what height would Philae have to drop on Earth to reach the same speed?

Useful numbers:

Earth's mass is 5.97×10^{24}kg.

Earth's radius is 6400 km.

London's latitude is 51ºN.

London and Orlando are around 7000 km apart.

Homework problems

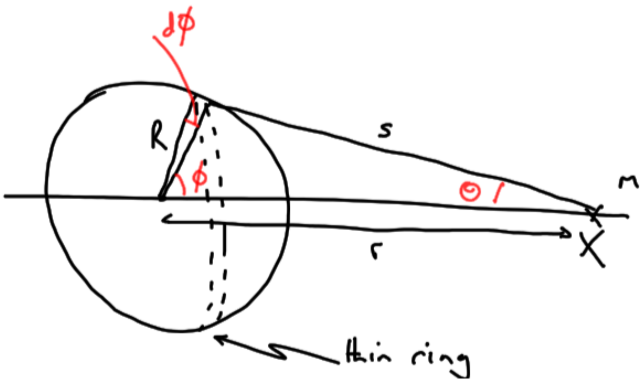
Attempt this in your own time; if you have problems, ask at your tutorial. It's quite a tricky one in places.

Newton's law of gravitation states that a body of mass m feels a gravitational force

$$F = -\frac{GMm}{r^2}$$

When it is a distance r from mass M. But what if M is not a point but an extended body, like a planet? When we stand on the Earth's surface, there is a bit very close to us, and some much farther away: can we *really* assume that this acts as a point mass at the centre of the Earth? Well, yes, it turns out that we can. This is what we will now prove – more precisely, that a spherical uniform mass acts, gravitationally, like a point mass at its centre.

We first consider a thin spherical shell of matter, of mass M. Consider the following diagram, where we are interested in the gravitational force on a mass m a distance r from the centre of the shell, at position X. We have marked a thin ring which is a distance s from X: by picking this arrangement, every point on the ring is the same distance from X. The ring has an angular width $d\phi$.



1. The thin ring has a mass dM. Show that the gravitational force on mass m due to the ring is

$$dF = \frac{GdMm}{s^2}\cos\theta$$

From where does the $\cos\theta$ factor arise?

2. Consider the area of the ring relative to the total area of the shell, and hence show that the fractional mass of the ring is

$$\frac{dM}{M} = \frac{1}{2}d\phi\sin\phi$$

3. In order to find the total force from the shell, we need to integrate over all the possible thin rings. Show that we can write this integral as

$$F = \frac{GMm}{2}\int_0^{\pi}\frac{\sin\phi\cos\theta}{s^2}d\phi$$

4. We want to turn this into an integral in s. Considering triangles in the diagram, show that we can write

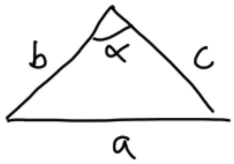
$$s^2 = (r - R\cos\phi)^2 + (R\sin\phi)^2$$

Differentiate this, remembering that r and R are fixed, to get

$$\sin\phi \, d\phi = \frac{s}{rR} ds.$$

5. A bit more tricky: the “law of cosines” says that for a triangle where one of the angles is α and the length of the opposite side is a, and the other two sides are b and c, then

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$



Hence show that we can write $\cos\theta = \frac{s^2+r^2-R^2}{2sr}$.

6. We’re nearly there. Substituting, show that we can write

$$F = \frac{GMm}{4}\frac{1}{r^2R} \int \left(1 + \frac{R^2 - r^2}{s^2}\right) ds$$

What are the correct limits on the integral?

7. Integrate to show

$$F = \frac{GMm}{r^2}$$

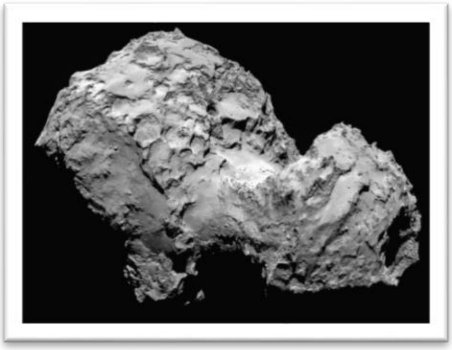
This is the expression we want: this shows that a spherical shell of material acts like a point mass at its centre. We can then consider the Earth as a series of concentric shells, and since each acts like a point mass, the total does the same.

If you want, you can do the integration over multiple shells to prove it to yourself – you could even show that, so long as each shell is homogeneous, the density can vary with depth and the body will still act like a point mass.

Another result you can prove is that *inside* an homogeneous spherical shell, the gravitational force is zero.

Tutorial problem: why doesn't Churyumov-Gerasimenko fall apart?

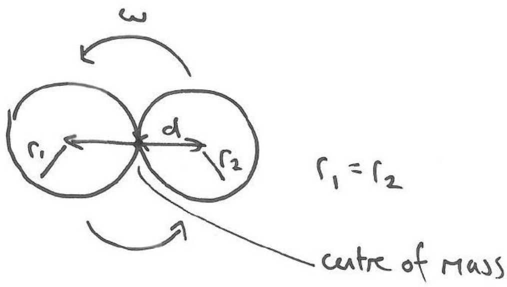
This is a problem that your tutor might choose to cover during the tutorial.



The Rosetta spacecraft, flying in formation with comet 67P/ Churyumov-Gerasimenko, has shown that the body, of total mass $M = 10^{13}\ kg$, is formed of two distinct parts. It rotates every 10 hours.

This is not the only comet or asteroid to be apparently formed of two parts. Comets are thought to be structurally weak objects: could they be spinning fast enough to fall apart?

1. The simplest case first: consider the comet as two uniform identical spheres. How fast must they be rotating about their centre of mass to touch, with no contact force? This is equivalent to saying that they are in circular orbits around each other at a distance apart of twice their radii:



Taking a density[1] of $\rho = 200 kg/m^3$, find the radius r of the spheres. By considering the acceleration in a circular orbit and being careful to distinguish between the orbital radius and d, the separation between the centres of the two spheres, show that

$$\omega = \sqrt{\frac{GM}{d^3}}$$

What do you get as the period: what are the implications for the comet? This is a very simple case, however, so let's make it a little bit more realistic and include unequal sized spheres. To do this we need to use the reduced mass.

[1] The estimated density is about twice this figure: we're using the smaller one because we're assuming spheres, whereas the real shapes are much less regular, so less space-filling. If you object to this assumption, feel free to try the numbers with the higher density.

2. Consider two masses at positions r_1 and r_2, experiencing no external forces, which act on each other with equal and opposite forces $F_{12} = -F_{21}$. Since we can write Newton's second law as
$$\frac{F_{12}}{m_1} = \frac{d^2 r_1}{dt^2}$$
And similarly for particle 2, show that
$$F_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2 r}{dt^2}$$
Where $r = r_1 - r_2$ is the separation vector between the two particles. Hence show that we can write
$$F_{12} = \mu \frac{d^2 r}{dt^2}$$
where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. So we can consider the relative motion of the two particles under the effect of their forces using the reduced mass. Note that if $m_1 \gg m_2$, we simplify to the problem of a single particle orbiting a fixed point.

3. Quick check: show that you get the same answer for equal sized spheres using the reduced mass equations as you did in the first part.

4. Finally, the more complex case: take one sphere to be twice the radius (but the same density) as the other. Try to be efficient as you calculate what is the critical rotational period for this case: a lot of things cancel. Are asymmetric bodies more, or less stable, than symmetric ones? What do you think is the fate of comet Churyumov-Gerasimenko?