Mechanics 2017

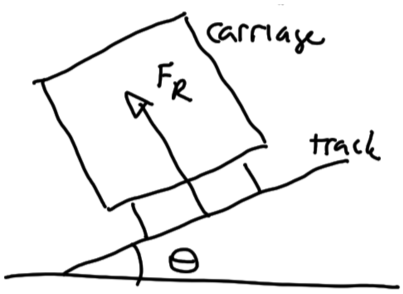
Problem sheet 8

1 December 2017

Assessed problem

You need to hand in your answers to this question. Write clearly and make sure you show your working.

The managers of a railway company are trying to improve the comfort of passengers on its trains coming into London. As the trains go round corners at speed, people feel as if they are pushed sideways, which is uncomfortable. It is therefore decided to tilt the track inwards at an angle θ, so the total force F_R people feel is directed downwards relative to the carriage.



(a) For motion at a constant speed v around a circular corner of radius r, write down an expression for the force that must be directed inwards on an object of mass M to maintain the motion.

[2 marks]

(b) In the case of the tilted track, the train travels at a speed such that the horizontal component of the force of the rails provides the force in part (a). By considering both horizontal and vertical components of the force on the train from the rails, show that the required tilt of the track is given by

$$\tan\theta = \frac{v^2}{rg}.$$

[4 marks]

(c) For the tight tracks in inner London, $r = 400m$. For a train to travel at 80 miles per hour, at what angle must the track be tilted? You should take $g = 9.8\text{ms}^{-2}$.

[2 marks]

(d) As a fraction of their original weight, how much "heavier" do passengers feel – that is, how much more force acts on them through their seat – compared to when the train is travelling on a straight track?

[2 marks]

Homework problems

Attempt these in your own time; if you have problems, ask at your tutorial.

1. Under a central force, angular momentum is conserved. For a given angular momentum $\underline{L} = \underline{r} \times \underline{p}$, when the velocity is at an angle θ to r, what is the speed perpendicular to r? Show, therefore, that Kepler's second law – that objects sweep out equal areas in equal time along their orbits – is true for all central forces.

2. In this question, you will determine some key properties of elliptical orbits, which are bound and so the total energy $E < 0$.

   (a) Recall that the energy equation for a gravitational orbit is

   $$E = \frac{1}{2} m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

   Closest approach to the Sun is called perihelion, r_p; the farthest point is aphelion, r_a. At both of these points, $\dot{r} = 0$. Show, therefore, that we can write

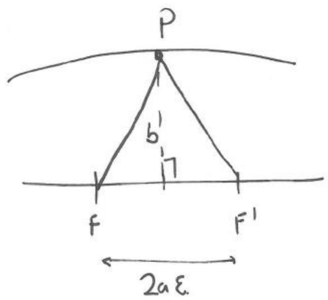
   $$r_p = a(1 - \epsilon)$$
   $$r_a = a(1 + \epsilon)$$

   Where the ellipse's semi-major axis $a = -GMm/2E$ and the eccentricity

   $$\epsilon = \sqrt{1 + \frac{2EL^2}{G^2 m^3 M^2}}$$

   Since $E < 0$, the semi-major axis is positive as we expect.

   (b) For an ellipse, the distance $F - P - F'$ (linking the particle to both foci) is a constant, $2a$. Consider the situation when the particle is equidistant from both foci:



   Remembering that $F - F' = 2a\epsilon$, by considering triangles, show that $b^2 = a^2(1 - \epsilon^2)$.

(c) Now substitute your expressions for a and ϵ and find:

$$b = \frac{L}{\sqrt{-2mE}}$$

So we find that the semi-major axis is just determined by the total energy of the orbit, while the semi-minor axis (and hence also the eccentricity) also contains the angular momentum L. The maximum angular momentum orbit for a given energy is a circle.

(d) Now we derive Kepler's third law, which states that the period of the orbit is related to its semi-major axis as $T^2 \propto a^3$. The area of an ellipse is $A = \pi ab$. In question 2, you calculated the rate of area swept out, dA/dt. We can therefore calculate the period of the orbit: $T = A/(dA/dt)$.

It is easier to calculate T^2. Substitute your expression for b and remember that $E = -GMm/2a$. You should find that

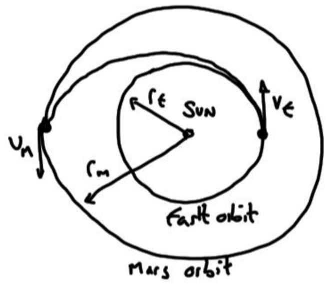
$$T^2 = \frac{4\pi^2}{GM} a^3$$

This proves Kepler's third law for elliptical orbits.

Tutorial problem: let’s go to Mars!

Many spacecraft have been sent from Earth to Mars; most of them made it and some of those even worked when they arrived.

A “Hohmann transfer” is generally the lowest thrust way to get from one planet to another and involves one rocket burn when leaving the vicinity of one planet and another when arriving at the second; the orbit in between is one whose perihelion is the closer planet and aphelion the farther. It’s easier to see in a diagram:



We will make several assumptions, some better than others: a rocket firing has zero duration; planets have circular and co-planar orbits; the gravitational effect of planets can be neglected. The Sun’s mass is 2.0×10^{30}kg.

1. The Earth orbits the Sun at 1 Astronomical Unit, $r_E = 150\times10^6$km. Given that it takes 365 days to orbit the Sun, calculate the orbital speed of the Earth. Mars is at $r_M = 1.522$ AU: use Kepler’s third law, or otherwise, to determine Mars’ orbital period and hence its speed.

2. The spacecraft performs a rocket burn when leaving the Earth and we assume that the change in velocity due to this burn is its departure velocity. In which direction should this burn be performed?

3. Determine the orbital velocity of the Hohmann transfer orbit at perihelion (the Earth). Hence calculate the required change in speed, δv. This is easiest to do using the *vis viva* equation and the semi-major axis.

4. What should the δv be when arriving at Mars, and in which direction, to make the spacecraft have zero relative speed with the planet? If the rocket engine has an exhaust speed $u = 2$km/s and the mass of the rocket and payload is 1 ton, how much fuel is required to get to Mars?

5. Can you think of another way of losing the kinetic energy on arrival? Would this save much fuel?

6. How long does the transfer orbit take? (Bonus question: could you sit in a small aluminium box for this long?).

7. Earth and Mars move relative to each other, so we need the right phasing of the two planets when we start on the transfer. How often do such alignments occur? This is why “launch windows” are so important for interplanetary spacecraft. Are launch windows more or less frequent for Jupiter ($r_J = 5.4$AU)?