Imperial College London

Department of Physics

Vibrations and Waves

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Lecture notes may be found on Blackboard (http://bb.imperial.ac.uk)

Chapter 1

Simple Oscillations

1.1 Introduction

Oscillatory systems are everywhere. Therefore, understanding quantities (x) that fluctuate in time (t), and learning how to characterise them and predict their behaviour is of fundamental importance in quantitative sciences. Most oscillatory systems are not simple harmonic oscillators - they are usually much more complex and interesting, requiring sophisticated analytical and numerical techniques to understand them.

In this course we will study relatively-simple oscillatory systems, most of which are mechanical. However, the concepts and methods introduced to study mechanical oscillations ("vibrations") are also applicable to non-mechanical oscillatory systems and form a basis for learning some of the more sophisticated methods later in your degree. Some examples of fluctuating quantities ($x(t)$) are

- the angle that a clock's pendulum makes with the vertical
- the charge on a capacitor when it is connected in series to an inductor
- the volume of your heart
- the position of an atom in a solid
- the populations of predators and their prey in an ecosystem
- the surface temperature of the Earth

We will start with the simplest of oscillatory systems: the simple harmonic oscillator (SHO). You may all be familiar with simple harmonic motion (SHM), but I will go through it from the beginning so that you're familiar with my notation.

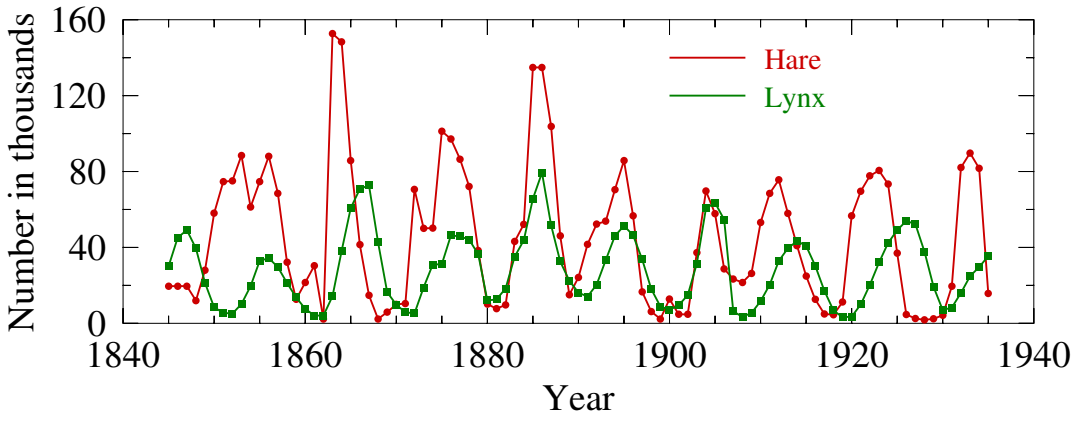


Figure 1.1: Changes in the abundances of the lynx and the snowshoe hare in Canada, as indicated by the numbers of pelts sold by hunters to the Hudson Bay Company (adapted from MacLuluch, 1937).

1.2 Stable equilibrium

A physical system is said to be at a **stable equilibrium** if, when it is perturbed slightly from that state, it returns to it. An example is an orange rolling in a bowl. The orange can remain at rest at the bottom of the bowl, which is a stable equilibrium position for it. If it is moved away from the bottom, it will feel a net force that moves it back towards the bottom. However, because it gains momentum as it rolls towards its equilibrium, it doesn't stop when it gets there. It overshoots it, whereupon it decelerates and reverses the direction of its motion. It begins oscillating about the bottom of the bowl until all of its energy has dissipated and it comes to rest at the bottom once more.

How long this takes depends on how quickly the energy dissipates. If the orange dissipates its energy quickly (e.g. via friction, sound, momentum transfer to the bowl, air resistance) the oscillations won't last very long. If the orange and the bowl were submerged in water the orange would feel strong resistance from the water and its energy would be transferred quickly to it. In this case, its motion would be heavily damped (excuse the pun!). In fact, depending on the orange, water might be so dense or viscous a fluid that the orange may not oscillate at all. Assuming that the orange sinks*, if the damping is heavy enough it

*Since writing, I discovered that oranges actually float unless their rind is very thin.

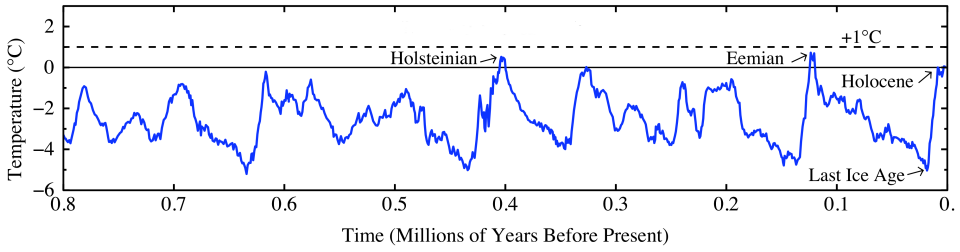


Figure 1.2: Global temperature, relative to the peak temperature in the Holocene epoch, versus time. (Source: James E. Hansen and Makiko Sato, NASA GISS: Science Briefs: Earth's Climate History: Implications for Tomorrow)

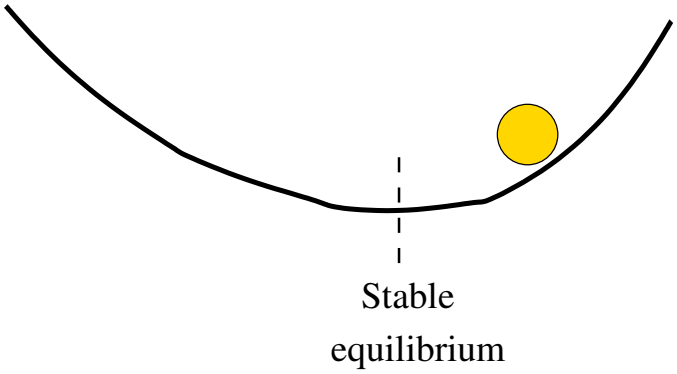


Figure 1.3: An orange rolling in a bowl has a stable equilibrium at the bottom of the bowl. It can remain there at rest indefinitely. If it is displaced from the bottom, it rolls back towards it.

would simply roll slowly to the bottom and settle there without oscillating.

If we were to study this oscillatory system and to develop a predictive mathematical description of its behaviour, it would be sensible to start with the most simplified model of its grossest features. We could gradually add more aspects of the system's behaviour into the model until we had a rather accurate and complete quantitative and qualitative description of it.

For example, we could consider a two dimensional orange and bowl, neglect energy dissipation and study the case of very small oscillations about the bottom of the bowl. The next step might be to see what happens in three dimensions or when we include energy dissipation. We might then consider larger oscillations, for which the shape of the bowl becomes more important. We might also want to consider what would happen if something external was influencing the orange's oscillations. For example, if the bowl was on a rocking boat, how would we describe that mathematically?

1.3 Simple Harmonic Oscillator

We'll begin the study of oscillations with the simplest kind of oscillatory behaviour, known as **simple harmonic motion** (SHM). As I'll explain, the term **"harmonic"** effectively means that oscillations are small enough in amplitude that the oscillator behaves in a very simple way.

As is customary, we'll use the example of a mass, m, on a spring. In this example, $x(t)$ is a displacement. It is the difference between the instantaneous length of the spring and its length when it is at equilibrium. According to **Hooke's law**, the magnitude of the restoring force of a stretched/compressed spring is proportional to the magnitude of the extension/compression. This is a good approximation when the extension/compression is small.

Mathematically, we write

$$F = -kx \tag{1.1}$$

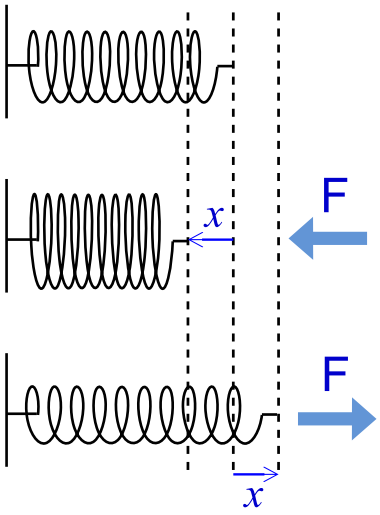
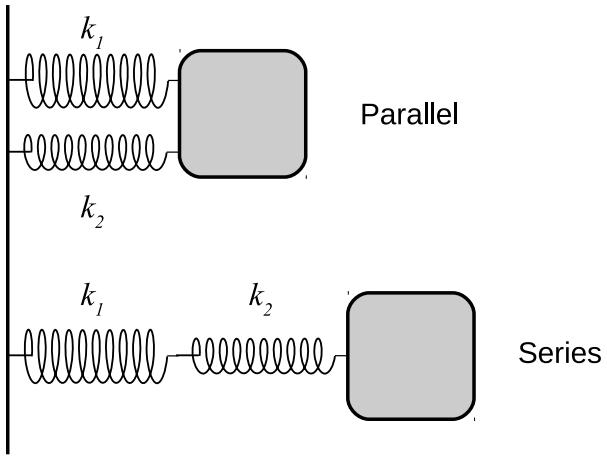
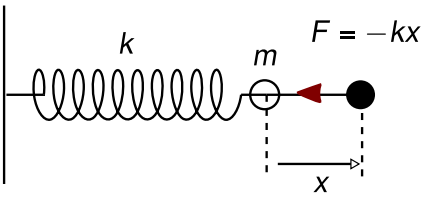


Figure 1.4: Hooke's law: The force required to compress/extend an ideal spring is proportional to the compression/extension. Extension/compression by an amount x requires a force $F = kx$, where the constant of proportionality, k, is known as the **spring constant** or **stiffness** and is a property of the spring. The force required to change the length of a spring is equal in magnitude, but opposite in direction, to the "restoring" force - the force with which it tries to return to its natural length.



Exercise: A mass is connected to a wall by two springs of spring constants k_1 and k_2. The springs can be in parallel or in series, as illustrated. In each case (series and parallel) show that the combined effect of the springs is equivalent to that of a single spring of spring constant k and derive an expression for k in terms of k_1 and k_2.



where x is the change in the length of the spring from its equilibrium length, F is the restoring force, and k is the proportionality constant, known as the **spring constant**. The minus sign is there because the restoring force is in the opposite direction to the displacement. When applied to the mass, the force causes an acceleration a. Using Newton's second law, we get

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = -kx \tag{1.2}$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0 \tag{1.3}$$

where t is time and, in the second line and henceforth, dots are used to denote time derivatives, i.e. $\dot{x} \equiv \frac{dx}{dt}$, $\ddot{x} \equiv \frac{d^2x}{dt^2}$.

A solution to Eq. 1.3, which we verify by substitution, is

$$x(t) = A\cos(\omega t + \phi) \tag{1.4}$$

where A, ϕ, and ω are constants whose meanings will be explained below.

Taking time derivatives of x gives expressions for the velocity, v, and acceleration, a.

$$v(t) \equiv \dot{x}(t) = -A\omega\sin(\omega t + \phi) \tag{1.5}$$

$$a(t) \equiv \dot{v}(t) = \ddot{x}(t) = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x(t) \tag{1.6}$$

$$\therefore F(t) = ma(t) = -m\omega^2 x(t) \tag{1.7}$$

which is the same as Eq. 1.3 if

$$k = m\omega^2 \tag{1.8}$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} \tag{1.9}$$

If we plot $x(t)$ as a function of time, t, the meanings of A, ω, and ϕ become clear.

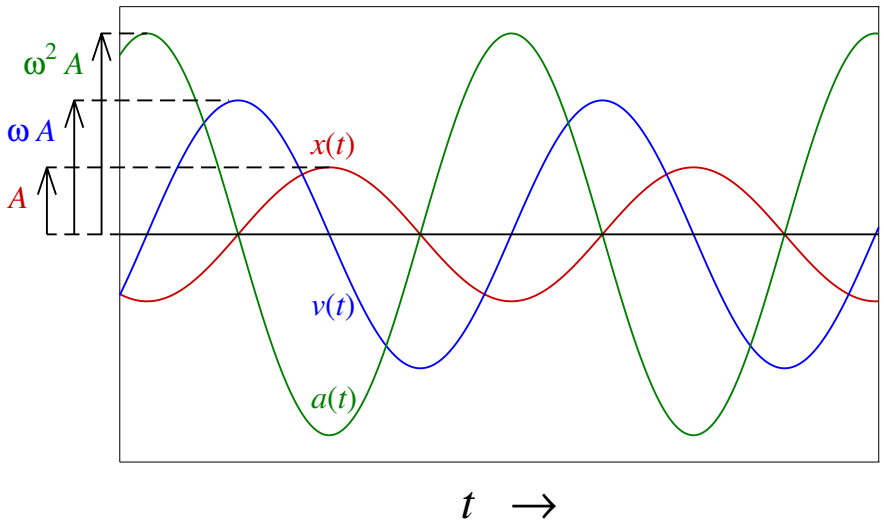


Figure 1.5: Plots of $x(t) = A\cos(\omega t + \phi)$, $v(t) = -\omega A \sin(\omega t + \phi)$, and $a(t) = -\omega^2 A \cos(\omega t + \phi)$. Note that acceleration is always in the opposite direction to displacement, but its magnitude is proportional to the displacement. Maximum speed ($|v(t)|$) is when displacement and acceleration are zero. Speed is zero when the magnitudes of displacement and acceleration are at their maxima.

Exercise:

Prove that

$$x(t) = A\sin(\omega t + \theta)$$

and

$$x(t) = B\cos\omega t + C\sin\omega t$$

are both mathematically equivalent to $x(t) = A\cos(\omega t + \phi)$, where θ, B, and C are constants.
Write down expressions relating ϕ to θ, A to B and C, and ϕ to B and C.

The maximum and minimum values of x are A and $-A$, respectively, where $A > 0$. A is called the **amplitude** of the oscillation. The motion repeats itself after a time T, known as its **period**. If, at any time t_0, its displacement and velocity are $x(t_0)$ and $v(t_0)$, respectively, this means that at all times $t_m = t_0 + mT$, where $m = 0, \pm1, \pm2, \cdots$, it is in precisely the same state: $x(t_m) = x(t_0)$, $v(t_m) = x(t_0)$.

The value of T depends on the argument of $\cos$ and $\sin$ in the expressions for $x(t)$ and $v(t)$. This time-dependent angle, $\omega t + \phi$, is known as the **phase** of the oscillation. Both functions repeat when the phase changes by an integer ($m \in \mathbb{Z}$) multiple of 2π, i.e., $\omega t_m + \phi = \omega t_0 + \phi + 2m\pi \Rightarrow t_0 - t_m = 2m\pi/\omega$. The period is the shortest time interval ($m =$

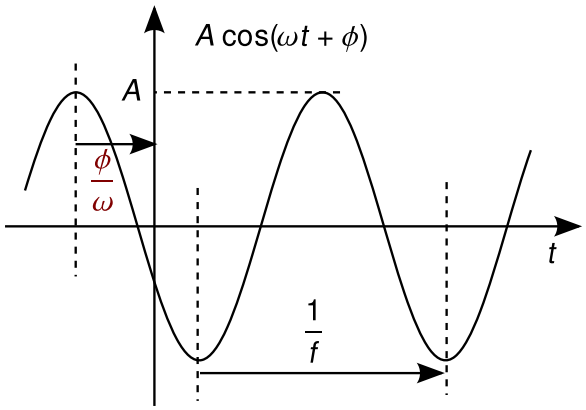


Figure 1.6: A plot of $x(t) = A\cos(\omega t + \phi)$ showing the amplitude (A), period ($T = 1/f$) and the shift backwards in time (ϕ/ω) relative to $A\cos(\omega t)$.

1) after which the system returns to the same state.

$$T = \frac{2\pi}{\omega} \tag{1.10}$$

The number of complete oscillations in a time interval Δt is $n = \Delta t/T$, and so the rate at which oscillations occur in that interval is $n/\Delta t = 1/T$. We refer to this as the **frequency**, f, of oscillations

$$f \equiv \frac{1}{T} \tag{1.11}$$

The frequency is the number of complete oscillations per unit time and it does not need to be a whole number. The SI units of f are s^{-1}, which is given the name "**hertz**" and the symbol Hz*. The $H - H$ bond length of a hydrogen molecule (H_2) oscillates with a period of approximately 8 fs = 8×10^{-15} s. Therefore, its frequency is $f = 1/(8 \times 10^{-15})$ Hz = 1.25×10^{14} Hz = 125 THz.

It is important to remember the relationships between f, T, and ω, and to understand their meanings.

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \text{period} = \text{duration of one complete oscillation}$$
$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \text{frequency} = \text{number of oscillations per unit time}$$
$$\omega = 2\pi f = \frac{2\pi}{T} = \text{angular frequency}$$

ω is known as the **angular frequency** because the argument of cos in Eq. 1.3 (the phase) is an angle expressed in radians†. The rate of change of the phase, in radians, with respect to time is ω. Putting this another way, ω is the rate at which the phase changes by one radian.

*After German physicist Heinrich Hertz (1857-1894).

†If the phase was expressed in degrees, we'd have to change the relationship between f and ω to $\omega = 360f$. Please just work in radians.

1.3.1 Phase constant

The constant ϕ doesn't change the frequency or the amplitude of the motion. As shown in Figure 1.6, it simply shifts the graph of $x(t)$ in time by $-\phi/\omega$. We often refer to ϕ as the **phase constant** of the oscillation.

Note that phase is a relative quantity and is only defined modulo 2π. If I change the phase $\omega t + \phi$ by $2n\pi$, where $n \in \mathbb{Z}$, I don't change it at all, effectively, because the periodicity of cos and sin is 2π. Usually we speak of the **phase difference** between two motions or dynamical quantities. For example, the phase difference between $\cos(\omega t + \phi)$ and $\cos(\omega t)$ is ϕ. The phase difference between $\cos(\omega t + \phi)$ and $\sin(\omega t)$, which is also a solution to Eq. 1.3, is $\phi + \pi/2$. The phase of $x_1(t) = A\cos(\omega t + \phi_1)$ *relative to* $x_2(t) = A\cos(\omega t + \phi_2)$ is $\Delta\phi = \phi_1 - \phi_2$. If I *decrease* ϕ_1 by an amount $\Delta\phi = \phi_1 - \phi_2$, I shift the graph of $x_1(t)$ forward

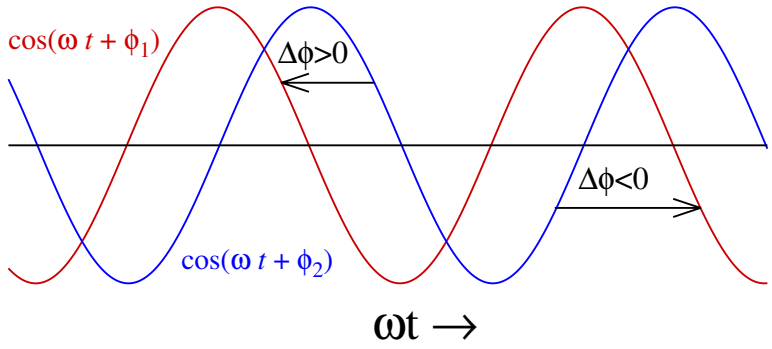


Figure 1.7: $x_1 = \cos(\omega t + \phi_1)$ and $x_2 = \cos(\omega t + \phi_2)$ plotted against ωt. I can think of x_1 as being shifted either forward or backward relative to x_2. $\Delta\phi = \phi_1 - \phi_2$ can be chosen to be positive or negative because both ϕ_1 and ϕ_2 are only defined up to an integer multiple of 2π. Therefore, $\Delta\phi \equiv \Delta\phi + 2m\pi$, where $m \in \mathbb{Z}$. If, by suitable choice of m, $\Delta\phi > 0$, then we usually think of x_1 as being shifted backwards in time relative to x_2. If $\Delta\phi < 0$, we usually think of it as being shifted forwards in time.

so that it coincides with $x_2(t)$. If I increased ϕ_1 I would shift it backward.

When $\phi_1 = \phi_2$ we say that x_1 and x_2 are **in phase**. When $\phi_1 \neq \phi_2$ we say that they are **out of phase** or that they are **phase shifted** relative to one another. When the phase difference between x_1 and x_2 is π, or an odd multiple of it, they are said to be in **antiphase**. If two sinusoids have the same amplitude and are in antiphase, their sum is zero:

$$x_1(t) + x_2(t) = A\cos(\omega t + \pi) + A\cos\omega t = -A\cos\omega t + A\cos\omega t = 0 \qquad (1.12)$$

1.3.2 Initial conditions

What determine's the phase constant and how do we know what the amplitude is?

As mentioned already, if I change the phase constant I shift $x(t)$ in time, which amounts to changing the arbitrarily-chosen origin of time. Therefore ϕ simply tells us the state of the system at $t = 0$.

A second-order differential equation, such as Eq. 1.3, can only be solved up to two unknown integration constants (A and ϕ) unless two further pieces of information are provided. These

are called the **initial conditions** and they tell us the state of the system at some point in time (often designated $t = 0$), or the partial state of the system at two different points in time.

The state of the system at time t is completely specified by $x(t)$ and $v(t)$. For example, acceleration is determined by displacement ($a = -\omega^2 x$). If we know x and v at any particular time, we can find A and ϕ and use them to find x and v at any other time.

We don't need to know x and v at the same instant to find A and ϕ. We could be told that

$$x(t_1) = \alpha$$
$$v(t_2) = \beta$$

Then we simply plug these values into our expressions for $x(t)$ and $v(t)$ to give two equations with our two unknowns: A and ϕ.

$$\alpha = A\cos(\omega t_1 + \phi) = A\cos\omega t_1 \sin\phi + A\sin\omega t_1 \cos\phi$$
$$\beta = -\omega A\sin(\omega t_2 + \phi) = -\omega A\sin\omega t_2 \cos\phi - \omega A\cos\omega t_2 \sin\phi$$

If we know $\omega = \sqrt{k/m}$, we can solve for A and ϕ.

For example, if we are told that

$$x(0) = 3.0 \text{ m}$$
$$v(0) = 0$$

then

$$\begin{aligned} 3.0 &= A\cos\phi \\ 0 &= -\omega A\sin\phi \\ \Rightarrow \phi &= 0\ (\pm 2m\pi) \quad \text{and} \quad A = 3.0 \\ \therefore x(t) &= 3.0\cos\omega t \end{aligned}$$

We could also be told the values of x at two different times, or the values of v at two different times. As long as we have two different* pieces of information we can deduce the two constants.

1.4 Circular motion, angular frequency, and phase

Consider a particle moving at constant angular velocity, ω, around the origin at a constant distance A from it, such that it sweeps out an angle ωt in a time t. Suppose that, as shown

*By "*different*" I mean that each piece of information teaches us something new and different about the oscillation. If we know that the period is $T = 2$ s, the values of $x(2.5)$ and $x(4.5)$ cannot be considered different pieces of information because, having been told $x(2.5)$, we already know the value of $x(4.5) = x(2.5 + T) = x(2.5)$.

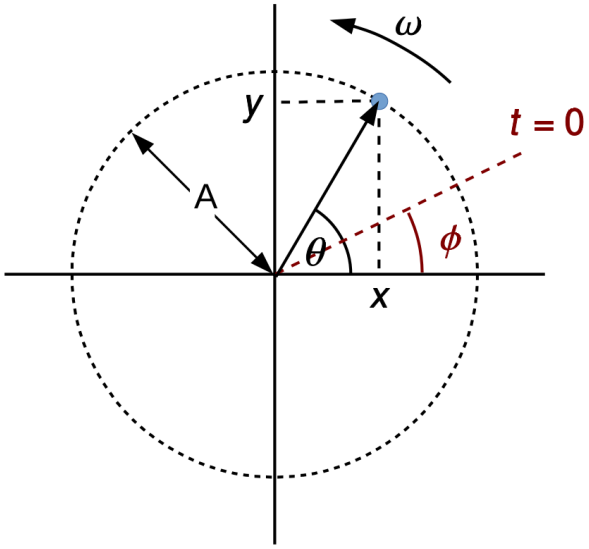


Figure 1.8:

in Figure 1.8, at $t = 0$ its displacement vector makes an angle ϕ with the positive direction of the $x-$axis and that it moves anti-clockwise. If $\vec{\imath}$ and $\vec{\jmath}$ are unit vectors pointing along the $x-$ and $y-$axes, respectively, the displacement of the particle from the origin when it is at an angle θ is*

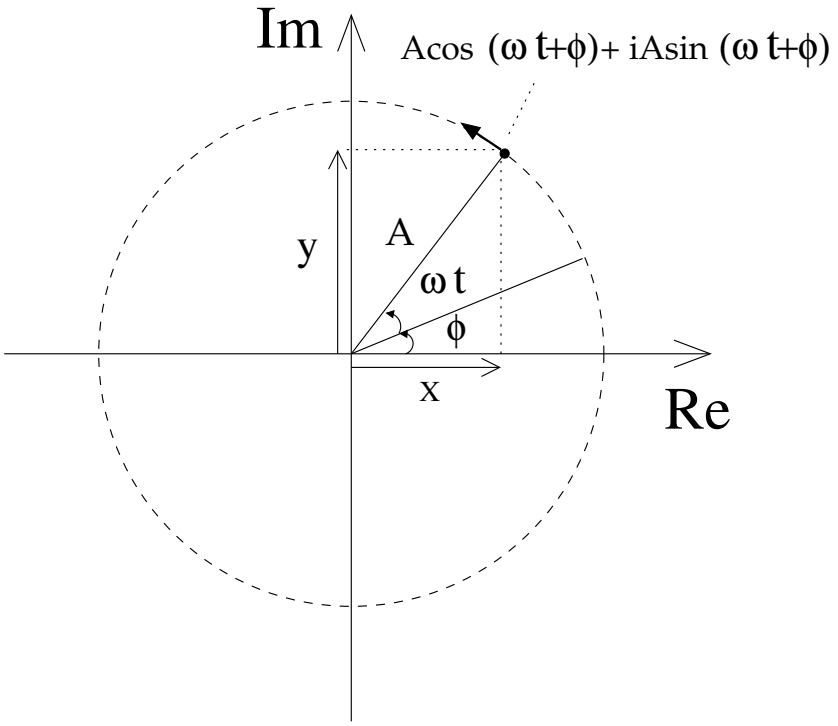
$$\vec{s} = x\vec{\imath} + y\vec{\jmath} = A\cos\theta\vec{\imath} + A\sin\theta\vec{\jmath} \tag{1.13}$$

At a time t, it is at angle $\theta(t) = \omega t + \phi$ and its displacement is

$$\vec{s}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} = A\cos(\omega t + \phi)\vec{\imath} + A\sin(\omega t + \phi)\vec{\jmath} \tag{1.14}$$

Note that $x(t)$ is identical to Eq. 1.4. Therefore, the $x-$component of the displacement of a particle in uniform circular motion around the origin has an identical time dependence to the displacement of a simple harmonic oscillator. The time taken for the particle to travel around the circle and return to its original position is equal to the time taken for it to sweep out an angle of 2π. It is $T = 2\pi/\omega$. The number of times the particle goes around the complete circle in a unit of time is $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$.

Note that the $y-$component of the displacement during circular motion is also a solution to Eq. 1.3. It corresponds to different initial conditions. When $y(t)$ is maximum, $x(t)$ is minimum and vice-versa.



1.5 Complex Notation

The relationship between circular and simple harmonic motions suggests a more convenient way to deal mathematically with oscillatory motion. Consider the complex number $\tilde{x} = x + iy$, where x and y are real numbers. We can plot $\tilde{x}$ on an Argand diagram and then imagine what happens if x and y vary continuously with time.

The time dependence of $\tilde{x}(t) = x(t) + iy(t)$ means that it moves in the complex plane. Let us write $\tilde{x}$ in polar form

$$\tilde{x}(t) = |\tilde{x}(t)|e^{i\theta(t)} \tag{1.15}$$

where $|\cdot|$ denotes modulus and $\theta(t)$ is the time-dependent phase.

From the previous section we know that, if $\tilde{x}$ were to move around the complex plane at a fixed distance from the origin (fixed modulus, $|\tilde{x}|$), and with a phase that changed at a constant rate $\omega = \dfrac{d\theta}{dt}$, both its real part, $x(t)$, and its imaginary part, $y(t)$, would be solutions to Eq. 1.3.

If $\theta(0) = \phi$ and $|\tilde{x}| = A$ then

$$\tilde{x}(t) = A\cos(\omega t + \phi) + iA\sin(\omega t + \phi) \tag{1.16}$$

$$\Rightarrow \tilde{x}(t) = Ae^{i(\omega t+\phi)} \tag{1.17}$$

where we have used Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. Now, taking time derivatives of $\tilde{x}$

*We will sometimes use the alternative notation for vectors $\vec{s} = (x, y)$ or $\vec{s} = \begin{pmatrix} x \\ y \end{pmatrix}$.

we find

$$\dot{\tilde{x}} = i\omega A e^{i(\omega t+\phi)} \tag{1.18}$$

$$\ddot{\tilde{x}} = -\omega^2 A e^{i(\omega t+\phi)} = -\omega^2 \tilde{x} \tag{1.19}$$

$$\Rightarrow \ddot{\tilde{x}} + \omega^2 \tilde{x} = 0 \tag{1.20}$$

For the left hand side of Eq. 1.20 to vanish, both its real part ($x = \mathrm{Re}(\tilde{x})$) and its imaginary part ($y = \mathrm{Im}(\tilde{x})$) must be zero. Therefore

$$\ddot{x} + \omega^2 x = 0 \tag{1.21}$$

$$\ddot{y} + \omega^2 y = 0 \tag{1.22}$$

So, instead of solving Eq. 1.3 to find $x(t)$, we could solve Eq. 1.20 to find $\tilde{x}(t)$ and then take either its real part or its imaginary part as the physical solution we are looking for. If one of our initial conditions was $x(0) = 0$, our solution would be $x(t) = \pm A \sin \omega t$ and it would be easier to take the imaginary part as our solution with $\phi = 0$. In this case we could also choose to take the real part as our solution, in which case ϕ would either be $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Using a complex exponential instead of trigonometric functions doesn't help us very much in this simple case. However, when we are dealing with more complicated oscillatory behaviour (e.g. damped oscillations, coupled oscillations) the use of complex exponentials can simplify the mathematics considerably.

Exercise:

Show that if $\tilde{x}(t) = x(t) + iy(t)$ is a solution to the differential equation

$$a\frac{d^2\tilde{x}}{dt^2} + b\frac{d\tilde{x}}{dt} + c\tilde{x} = f(t) \tag{1.23}$$

where a, b, c, $x(t)$, and $y(t)$ are all real valued, then

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = \mathrm{Re}\,(f(t)) \tag{1.24}$$

and (1.25)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \mathrm{Im}\,(f(t)) \tag{1.26}$$

where $\mathrm{Re}()$ and $\mathrm{Im}()$ refer to the real and imaginary parts, respectively.

1.6 Potential and kinetic energy

If nothing interferes with the harmonic oscillator its energy must be conserved. Of course, in reality, oscillations are always damped by friction or air resistance or something. But let's assume, for the moment, that the oscillator is completely isolated and conserves energy.

When $x = \pm A$ its velocity is zero (see Figure 1.5). Therefore, its kinetic energy is zero at these displacements and all its energy is stored in the spring as potential energy. When $x = 0$, the spring is unstretched and so it isn't storing any potential energy. When this is the case, all of the oscillator's energy is in kinetic form. Total energy (E) is conserved and so the maximum of kinetic energy ($K_{\text{max}} = E$) must equal the maximum of potential energy ($V_{\text{max}} = E$) because each is zero when the other is at its maximum. When $x \neq \pm A$ and $x \neq 0$, the oscillator has both kinetic and potential energy and their sum is $E = V_{\text{max}} = K_{\text{max}}$. Therefore, the SHO's energy flows back and forth between kinetic and potential forms during its motion.

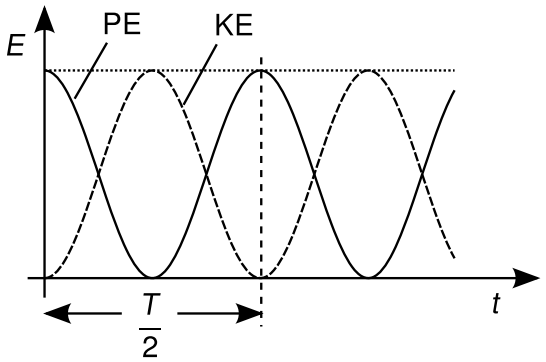


Figure 1.9: Potential energy (PE = $\frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t)$) and kinetic energy (KE = $\frac{1}{2}mv^2 = \frac{1}{2}kA^2\sin^2(\omega t)$) and their sum (E = PE+KE = $\frac{1}{2}kA^2$) as functions of time during simple harmonic motion.

The work done to extend the spring equals the change in potential energy of the spring. An infinitesimal change in extension, $x \to x + dx$, causes an infinitesimal change dV in the potential energy. From Hooke's law, the force is kx and so the work done is $kxdx$.

$$dV = kxdx \tag{1.27}$$

$$\Rightarrow \int_{V(0)}^{V(x)} dV = \int_0^x kx'dx' \tag{1.28}$$

$$\Rightarrow V(x) - V(0) = \frac{1}{2}kx^2 \tag{1.29}$$

Since $V(0) = 0$ (spring is unstretched at $x = 0$) we simply find that $V(x) = \frac{1}{2}kx^2$.

The maximum magnitude of the displacement is A and the velocity is zero when $x = \pm A$. Since the kinetic energy is zero when $x = \pm A$, the constant total energy of the oscillator equals potential energy at $x = \pm A$.

$$E = V_{\text{max}} = \frac{1}{2}kA^2 \tag{1.30}$$

$$\Rightarrow K_{\text{max}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kA^2 \tag{1.31}$$

$$\Rightarrow v_{\text{max}} = v(A) = A\sqrt{\frac{k}{m}} \tag{1.32}$$

We often like to express the energy in terms of the amplitude, (angular) frequency, and mass of the oscillator

$$E = \frac{1}{2}m\omega^2A^2 \tag{1.33}$$

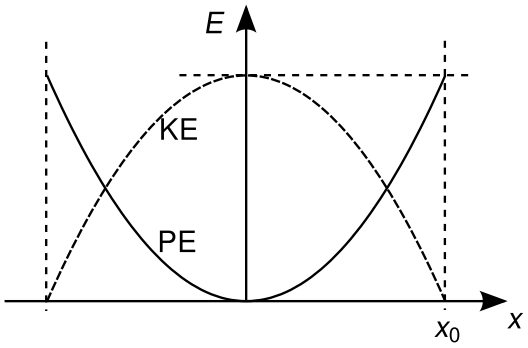


Figure 1.10: Potential energy (PE) and kinetic energy (KE) as functions of displacement from equilibrium (x) during simple harmonic motion.

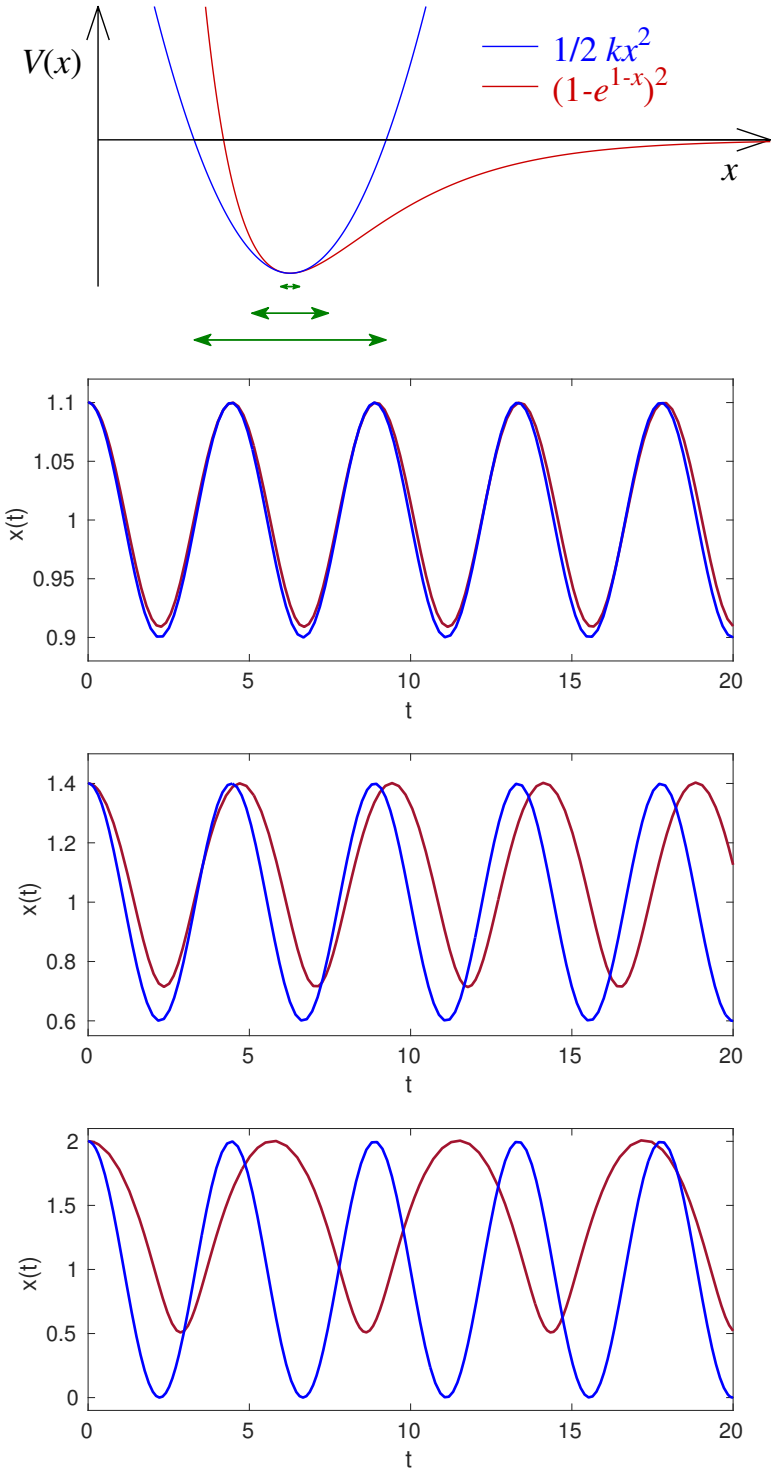
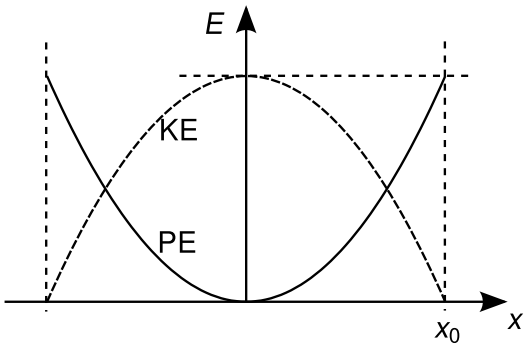


Figure 1.11: Displacement (x) as a function of time for oscillations of three different amplitudes in the harmonic potential $V(x) = x^2$ compared to oscillations of similar amplitude in the anharmonic "Morse" potential, $V(x) = (1 - e^{1-x})^2$, which resembles the interaction energy between two covalently bonded atoms. When the amplitude is very small, the oscillations in the morse potential are very similar to the simple harmonic oscillations in the parabolic potential. Deviations from simple harmonic behaviour increase as amplitude increases because more of the potential well is visited during each period and the period/frequency changes with amplitude. As a consequence of anharmonicity, both the frequency of the oscillation and the average bond length $\langle x \rangle$ depend on amplitude of the vibration. Anharmonicity is the cause of **thermal expansion**: high temperature $\Longrightarrow$ more energy per atom $\Longrightarrow$ larger amplitude vibration $\Longrightarrow$ larger average bond length.



1.7 What does "harmonic" mean?

Harmony:

- *the state of being in agreement or concord*
- *a pleasing combination of elements; a pleasing layering or arrangement of sounds*

If the potential energy is a quadratic function of x, the system is termed "harmonic" and the time dependence of x is sinusoidal. If there are any higher powers of x in the expression for the potential energy they are known as "anharmonic" terms. Anharmonic terms become increasingly important as the amplitude of an oscillation increases and they make $x(t)$ deviate from sinusoidal behaviour.

The reason for the term "harmonic" will become clearer later in the course when we study coupled oscillations and waves. Briefly, anharmonic oscillations can be thought of as oscillations which are not sinusoidal. When you study Fourier series later in your degree course you will learn that anharmonic oscillations consist of the superposition (sum) of multiple (often an infinite number) of harmonic oscillations with different frequencies.

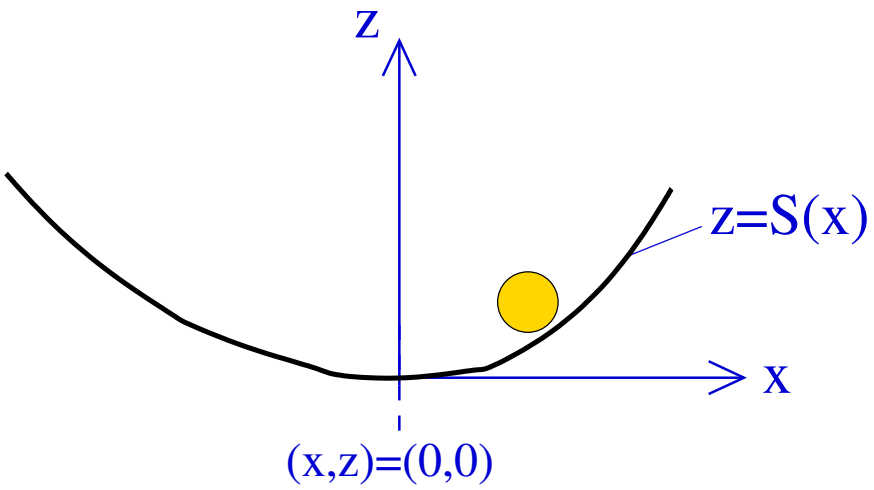
The purity of the sound of a vibrating string (e.g. guitar, violin, or piano string) depends on how closely its potential energy can be approximated as being quadratic in the amount that it is stretched. The potential energy of a *linearly elastic* (tension $\propto$ extension) string is perfectly quadratic. We'll discover later in the course that this means that the sound that it makes consists only of a dominant "fundamental" frequency, f, and integer multiples of this frequency, nf ($n \in \mathbb{Z}$) which are known as "harmonics".

For the musicians among you: an octave is a factor of two in frequency. This means that if you pluck an ideal (linearly elastic) string that has been tuned to $B\,\flat$, its first harmonic ($n = 1$) is $B\,\flat$, and its second ($n = 2$), fourth ($n = 4 = 2^2$) and eighth ($n = 8 = 2^3$) harmonics are also $B\,\flat$ but one, two, and three octaves above, respectively. The other harmonics ($n = 3, 5, 6, 7, 9, \cdots$) are not $B\,\flat$, but most are very close to notes on the scale. Furthermore, higher harmonics are not usually as loud as the lower ones *.

*In Western music there are 12 notes per octave which are separated from one another by a constant

The sound produced by a string that is not linearly elastic can contain an inharmonious mix of frequencies. Such a string can sound nasty/dissonant because its frequencies are not multiples of one another.

1.8 Why are small oscillations harmonic?



Let's consider the example of the orange rolling in a bowl and neglect energy dissipation. For simplicity, let's pretend that the orange and bowl are two dimensional and so, if we know the shape of the bowl, we can specify the position of the orange with a single displacement variable x, which is its lateral displacement from equilibrium. The bowl's shape is specified by the function $z = S(x)$, which is the height above the bottom of the bowl.

For displacement x, the orange's gravitational potential energy is greater than when it is at equilibrium by an amount $V(x) = mgz = mgS(x)$, where m is the mass of the orange, $g \approx 9.8$ ms^{-1} is the acceleration due to gravity.

Now let's use a Maclaurin/Taylor series expansion about the point $x = 0$ to find an expression for $V(x)$ in terms of x and the derivatives of $S(x)$ evaluated at $x = 0$.

$$V(x) = \cancelto{0}{V(0)} + \cancelto{0}{V^{(1)}(0)x} + \frac{1}{2!}V^{(2)}(0)x^2 + \frac{1}{3!}V^{(3)}(0)x^3 + \cdots + \frac{1}{n!}V^{(n)}(0)x^n + \cdots \tag{1.34}$$

$$= \frac{1}{2}V^{(2)}(0)x^2 + \frac{1}{6}V^{(3)}(0)x^3 + \frac{1}{24}V^{(4)}(0)x^4 + \cdots + \frac{1}{n!}V^{(n)}(0)x^n + \cdots \tag{1.35}$$

$$= \frac{1}{2}mgS^{(2)}(0)x^2 + \frac{1}{6}mgS^{(3)}(0)x^3 + \frac{1}{24}mgS^{(4)}(0)x^4 + \cdots \tag{1.36}$$

The first two terms in the Maclaurin series vanish because $y = S(x) = 0$ at $x = 0$ and the curve is flat (zero slope) at the minimum $\Rightarrow S^{(1)}(0) = 0$. The derivatives evaluated at $x = 0$ are just numbers: they are constants defining the bowl's shape near the minimum. Therefore, we have written the potential energy as an infinite power series in x with constant coefficients, c_n, and a lowest order term of $\sim x^2$

$$V(x) = c_2x^2 + c_3x^3 + c_4x^4 + \cdots \tag{1.37}$$

factor, r, where $r^{12} = 2$. This is a compromise between the constraints of musical instruments design and the desire to be able to change the key of a piece of music without changing the intervals between the notes.

It is always possible to find a value of x that is sufficiently small to make the first term dominate this expression. This is because, if $|x| \ll 1$, then $|x^2| \gg |x^3| \gg |x^4| \gg \cdots$. Therefore, for very small x the approximation $V(x) \approx c_2 x^2$ is a good one, and it becomes exact in the limit $x \to 0$.

1.8.1 Linear response

Recall that, for simple harmonic motion we found that $V(x) = \frac{1}{2}kx^2$. Comparing this to Eq. 1.35 we find that, in the limit $x \to 0$, the motion of the orange in the bowl is simple harmonic, with

$$k = m\omega^2 = V^{(2)}(0) = \left(\frac{d^2V}{dx^2}\right)_{x=0} \tag{1.38}$$

So, without knowing anything about the shape of the bowl, except that it is smooth, we have shown that very small oscillations about its bottom are simple harmonic.

We used Hooke's Law to prove that $V(x) = \frac{1}{2}kx^2$ for the harmonic oscillator. However, the reason that $F \propto -x$ should now be clearer. It is simply a consequence of the potential energy varying smoothly with extension, x, and $|x|$ being small enough to make the approximation $V(x) \propto x^2$ a good one.

Hooke's Law is an example of an assumption, which we make throughout this lecture course, known as *linear response*. Sometimes this assumption is justified, sometimes it less well justified, and sometimes it breaks down completely. Whether or not it is justified, you should always be aware that you are making it. So what is it?

If a physical system responds linearly, this means that the magnitude of its response R is proportional to the magnitude of the stimulus S creating it.

$$R \propto S \Rightarrow R = KS \tag{1.39}$$

Hooke's Law for springs says that the response to a force F is a displacement $x \propto F$. If the response was the polarization of a dielectric material (P) and the stimulus was an electric field (E), the linear response assumption is $P \propto E \Rightarrow P = \varepsilon_0 \chi_e E$, where $\varepsilon_0 \chi_e$ is a constant. Another example of linear response is the current (I) that flows across a resistor (of resistance R) in response to a small potential difference across it V): $I = \left(\frac{1}{R}\right) V$.

Linear response means that if I double the stimulus, I double the response. It means that if I decide to add an amount ΔS to the stimulus, I don't need to know what the existing stimulus (S) and response (R) are before determining what the resulting change in the response ΔR will be. It is $\Delta R = K\Delta S$ because $R = KS$ and $(R + \Delta R) = K(S + \Delta S)$

Quite generally, systems close enough to stable equilibrium respond linearly to stimuli. The only question is *how close is close enough?*

Note that *linearity* and *harmonicity* mean the same thing, for the purposes of this course. A harmonic system is one whose potential energy is quadratic ($V \sim x^2$) A linear system is

one whose force is linear ($F \sim x$). The force is the negative of the derivative of the potential energy ($F = -dV/dx$) and so both terms are equivalent. You may encounter a broader class of linear systems in other contexts, but not in this course.

1.8.2 Frequency, curvature, and softening of vibrations

Note that, because the equilibrium position of the orange in the bowl is a minimum (in height and potential energy)

$$\left(\frac{d^2S}{dx^2}\right)_{x=0} > 0 \Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=0} > 0 \Rightarrow k > 0 \tag{1.40}$$

Also note that

$$\omega^2 \propto \left(\frac{d^2V}{dx^2}\right)_{x=0} \propto \left(\frac{d^2S}{dx^2}\right)_{x=0}, \tag{1.41}$$

Therefore, as the curvature of the bowl reduces, the frequency reduces. When the bowl is completely flat (i.e. a plate rather than a bowl), then $\left(\frac{d^2V}{dx^2}\right)_{x=0} = 0 \Rightarrow \omega = 0$. The frequency is zero because the restoring force is zero. There is no oscillation and the orange will remain wherever you place it.

If we were to reduce k further, making it negative, the bowl would become inverted and $\omega^2 < 0 \Rightarrow \omega = \pm i|\omega|$. The frequency is an imaginary number! If our solution to the oscillator equation was $\tilde{x}(t) = Ae^{-i\omega t}$ and we reduced k until it became zero, and then became negative, we would get $\tilde{x}(t) = Ae^{|\omega|t}$. For $k < 0$, the displacement increases exponentially with time, because the orange simply rolls off the bowl, gathering pace as it goes.

When thinking of atoms in a crystal, it is often useful to call to mind a cartoon of atoms attached to one another by springs, which represent the bonds. Sometimes an external

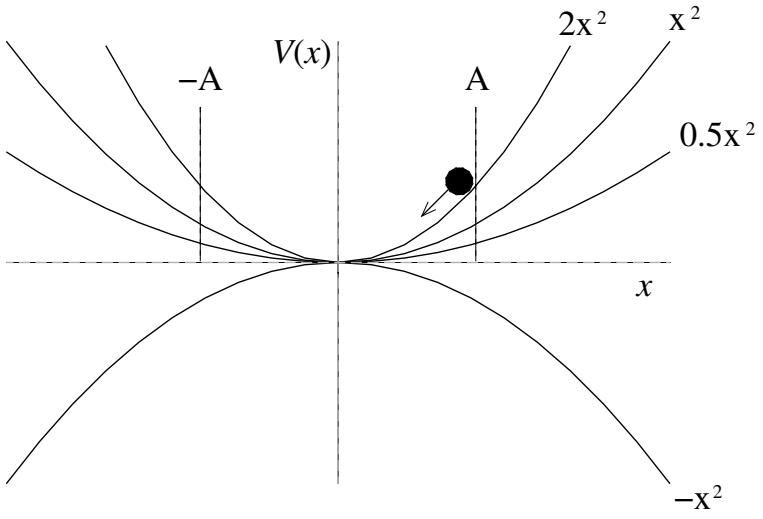


Figure 1.12: Plots of $V(x) = \frac{1}{2}kx^2$ for various values of k.

factor such as temperature or pressure weakens the bonds/springs and so the frequencies of the corresponding vibrations decrease ($\omega^2 \propto k$). We say that the crystal lattice vibration, which is known as a "phonon", *softens*. If k decreases past zero, so that it is negative, ω is imaginary and the atoms are no longer returned to their old equilibrium positions by the vibration. The consequence of $k < 0$ is that the crystal either melts or rearranges into a new and different crystal structure.

1.9 Appendix 1: Vector notation in one dimension

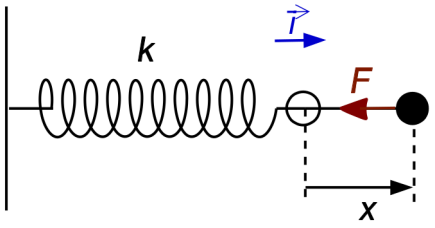


Figure 1.13:

Forces and displacements are vector quantities. When working in two or three dimensions we will usually use explicit vector notation, unless the problem at hand can be simplified to a one dimensional one. In one dimension, the direction of a vector is specified by its sign.

To make this clear, consider a mass resting on a frictionless surface and attached to a wall by a spring, as shown in Figure 1.13. We define a vector $\vec{\imath}$ of unit length ($|\vec{\imath}| = 1$), which is parallel to the spring, as shown. If $|\vec{x}|$ is the magnitude of the displacement $\vec{x}$ of the mass from equilibrium, and $|\vec{F}|$ is the magnitude of the restoring force $\vec{F}$ exerted by the spring, Hooke's law tells us that $|\vec{F}| = k|\vec{x}|$. When the displacement is in the $\vec{\imath}$–direction ($\vec{x} \cdot \vec{\imath} > 0$), we know that the force must be in the negative $\vec{\imath}$–direction. Therefore

$$|\vec{F}| = k|\vec{x}| \Rightarrow \vec{F} = k|\vec{x}|\left(-\vec{\imath}\right) = -k|\vec{x}|\vec{\imath} = -k\vec{x}$$

because $\quad \vec{F} = |\vec{F}|(-\vec{\imath}) \quad$ and $\quad \vec{x} = |\vec{x}|\vec{\imath}$

When the displacement is in the opposite direction ($\vec{x} \cdot \vec{\imath} < 0$, $\vec{F} \cdot \vec{\imath} > 0$),

$$|\vec{F}| = k|\vec{x}| \Rightarrow \vec{F} = k|\vec{x}|\vec{\imath} = -k|\vec{x}|\left(-\vec{\imath}\right) = -k\vec{x}$$

because $\quad \vec{F} = |\vec{F}|\vec{\imath} \quad$ and $\quad \vec{x} = |\vec{x}|\left(-\vec{\imath}\right)$

In both cases we get

$$\vec{F} = |\vec{F}|\vec{\imath} = -k|\vec{x}|\vec{\imath} = -k\vec{x} \tag{1.42}$$

We can simplify our notation by defining $F \equiv \vec{F} \cdot \vec{\imath}$ and $x \equiv \vec{x} \cdot \vec{\imath}$. Taking the scalar product of both sides of Equation 1.42 with $\vec{\imath}$ gives us

$$\vec{F} \cdot \vec{\imath} = -k\vec{x} \cdot \vec{\imath} \tag{1.43}$$

$$\Rightarrow F = -kx \tag{1.44}$$

The signs of F and x tell us their directions - whether they are parallel or antiparallel to $\vec{\imath}$. The magnitudes of F and x equal the magnitudes of $\vec{F}$ and $\vec{x}$, respectively.

So, in one dimension we can drop the explicit vector notation because only two directions are possible ($\vec{\imath}$ and $-\vec{\imath}$), corresponding to the two possible signs of our vector quantities.

Appendix 2: Taylor Series

The Taylor series expansion of a function $f(x)$ about a point $x = a$ is

$$\begin{aligned} f(x) &= f(a) + \left(\frac{df}{dx}\right)_{x=a}(x-a) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\right)_{x=a}(x-a)^2 \\ &\quad + \frac{1}{3!}\left(\frac{d^3f}{dx^3}\right)_{x=a}(x-a)^3 + \cdots + \frac{1}{n!}\left(\frac{d^nf}{dx^n}\right)_{x=a}(x-a)^n + \cdots \\ &= f(a) + f^{(1)}(a)(x-a) + \frac{1}{2!}f^{(2)}(a)(x-a)^2 + \cdots \frac{1}{n!}f^{(n)}(a)(x-a)^n + \cdots \end{aligned}$$

where, in the last line, I have used the shorthand notation $f^{(n)}(a) \equiv \left(\frac{d^nf}{dx^n}\right)_{x=a}$ for the n^{th} derivative at $x = a$.

The infinite Taylor series is an exact mathematical expression for $f(x)$ iff f and all its derivatives exist and are continuous between a and x. If x is close to a, a truncated Taylor series (e.g. $f(x) \approx f(a) + f^{(1)}(a)(x-a) + \frac{1}{2}f^{(2)}(a)(x-a)^2 + \frac{1}{3}f^{(3)}(a)(x-a)^3$) can often be a very good approximation to $f(x)$, with the inclusion of more terms in the series providing a better approximation.

A function of two variables, $g(x, y)$ can be expanded about the point $(x, y) = (a, b)$ as follows

$$\begin{aligned} g(x,y) &= g(a,b) + \left(\frac{\partial g}{\partial x}\right)_{(a,b)}(x-a) + \left(\frac{\partial g}{\partial y}\right)_{(a,b)}(y-b) \\ &\quad + \frac{1}{2!}\left(\frac{\partial^2 g}{\partial x^2}\right)_{(a,b)}(x-a)^2 + \frac{1}{2!}\left(\frac{\partial^2 g}{\partial y^2}\right)_{(a,b)}(y-b)^2 + \frac{1}{2!}\left(\frac{\partial^2 g}{\partial x \partial y}\right)_{(a,b)}(x-a)(y-b) \\ &\quad + \cdots + \frac{1}{(n_x+n_y)!}\left(\frac{\partial^{n_x+n_y} g}{\partial x^{n_x}\partial y^{n_y}}\right)_{(a,b)}(x-a)^{n_x}(y-b)^{n_y} + \cdots \end{aligned}$$

Chapter 2

Coupled Oscillations and Normal Modes

2.1 Introduction

Vibrations rarely exist in isolation. There is always some coupling to the mechanical system's environment. Sometimes this coupling is weak and dissipative (taking energy out of the system) and we can treat it as a damping force. We will deal with this situation in Chapter 3. Sometimes the coupling is to an immutable driving force. We will discuss this situation in Chapter 4. This chapter is about the coupling of an oscillator to another oscillator, or to several other oscillators, such that, in the absence of damping or an external driving force, the energy of the complete multi-oscillator system is conserved. The energy of each individual oscillator is not conserved because they exchange energy with one another: this is what "coupling" means.

To begin studying coupled oscillators, let us return to the example of the orange rolling in a bowl. Previously, we introduced the simplification that the orange and bowl were two dimensional and that the orange only moved in one dimension. Now let us return to three dimensions, using the coordinates (x, y, z). Let us assume that, when the orange is at the bottom of the bowl (its stable equilibrium), its coordinates are $(0, 0, 0)$ and that when it is at a general position (x, y, z), the value of z is the difference $h(x, y)$ between its height and its height at equilibrium, while (x, y) is the orange's lateral displacement (in the plane parallel to the table on which the bowl rests) from equilibrium.

We will assume that the orange always remains in contact with the bowl and so the height $h(x, y)$ is defined by the shape of the bowl. Because z is determined by x and y, effectively we only have to describe the dynamics in those two dimensions to have a full description in all three dimensions. Henceforth, I will specify its position as (x, y) and its equilibrium is at $(0, 0)$. The potential energy at (x, y) relative to the bottom of the bowl is $V(x, y) = m\,g\,h(x, y)$, where m is the orange's mass and g is the acceleration due to gravity*. If we place the orange at an arbitrary position (x, y), it will roll in the general direction

*Recall that in two dimensions, at a minimum of $V(x, y)$, we have

$$\left.\frac{\partial V}{\partial x}\right|_{(0,0)} = \left.\frac{\partial V}{\partial y}\right|_{(0,0)} = 0; \quad \frac{\partial^2 V}{\partial x \partial y} > 0, \quad \frac{\partial^2 V}{\partial x^2} > 0, \quad \frac{\partial^2 V}{\partial x^2} > 0$$

of the minimum, but it may not pass through the point $(0,0)$ unless the bowl has a very symmetric shape. For a bowl that is not circularly symmetric about $(0,0)$, it will roll around the minimum so that, although both x and y oscillate in some complex way about $x = 0$ and $y = 0$, respectively, it is unlikely they will never both be zero simultaneously until the the orange comes to rest. We will neglect dissipation in this chapter, so that can never happen.

A very important point to note is that, from a mathematical perspective, there is no difference between an oscillator of mass m in two dimensions and two coupled one dimensional oscillators, each having mass m^*. In each case we simply have a potential energy $V(x,y)$ which is a function of two variables - one for each degree of freedom, and Newton's second law applied to each degree of freedom separately gives

$$F_x(x,y) = m\ddot{x} = -\left.\frac{\partial V}{\partial x}\right|_{(x,y)}$$
$$F_y(x,y) = m\ddot{y} = -\left.\frac{\partial V}{\partial y}\right|_{(x,y)}$$

More generally, N degrees of freedom is equivalent, mathematically, to N dimensions. Therefore, everything I am telling you about the orange rolling in the bowl applies to two coupled oscillators. For the orange moving in two dimensions, x and y are two orthogonal ($\perp$) components of its displacement from the equilibrium state $(x,y) = (0,0)$. For two coupled one dimensional oscillators, x is the displacement of one oscillator from its position *when the coupled system of two oscillators is at equilibrium* and y is the displacement of the other from its position when the coupled system is at equilibrium.

In general, for coupled oscillators, or for the orange rolling in the bowl,

$$F_x(0,y) = -\left.\frac{\partial V}{\partial x}\right|_{(0,y)} \neq 0 \text{ and } F_y(x,0) = -\left.\frac{\partial V}{\partial y}\right|_{(x,0)} \neq 0$$

This means that neither x nor y can be at equilibrium unless both are simultaneously at equilibrium. V depends on *both* x and y and so $\ddot{x}(t)$ depends on the $y(t)$ and $\ddot{y}(t)$ depends on $x(t)$. This coupling can lead to motion that looks horribly complex.

*Things are only very slightly more complicated when the coupled oscillators have different masses.

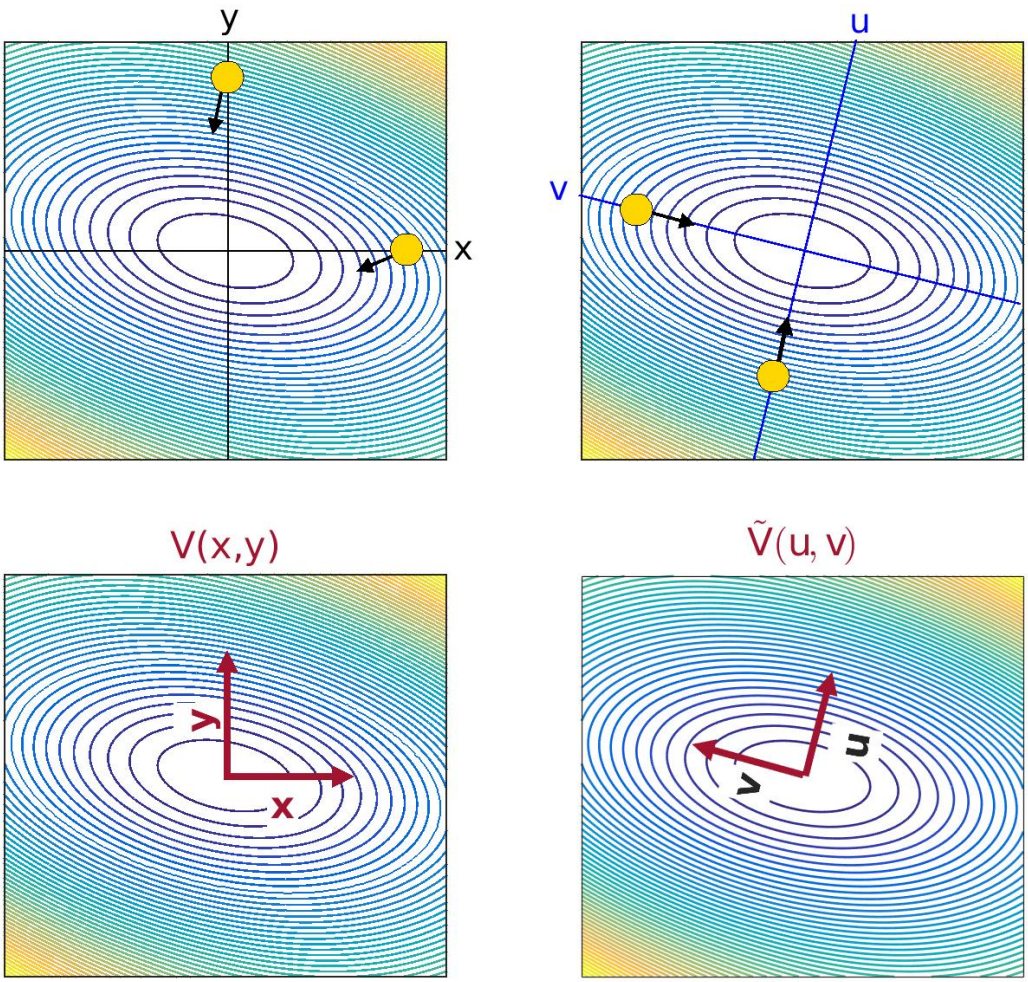


Figure 2.1: Contour plots of the potential energy. The contours (lines) connect points of equal potential energy (equal height, for the orange in the bowl). The force at any point, is perpendicular to the contours at that point because that is the direction of steepest descent. At any position along the $x-$axis the force has both an $x-$component and a $y-$component (top left panel). The same is true at any position along the $y-$axis. However, there exists another set of axes, u and v, such that, at each position along the $u-$axis the $v-$component of the force is zero and at each position along the $v-$axis, the $u-$component of the force is zero (top right and bottom right). This is only true close to the minimum where the potential is harmonic. If the orange is placed on the $u-$axis ($v = 0$) it will execute simple harmonic motion along this axis and v will remain zero. If it is placed on the $v-$axis, it will execute SHM along this axis and u will remain zero. If it is placed at an arbitrary position its motion will not be simple harmonic and both u and v will change with time. However, the motion can be expressed mathematically as the sum of two simple harmonic motions - one parallel to the $u-$axis and the other parallel to the $v-$axis. Note that when we look at the $u-$axis in the xy coordinate system, as shown in the top right panel, it is just a straight line passing through the origin. Therefore, its equation is $y = r_u x$ where r_u is its slope. This means that, when $v = 0$ and the orange is oscillating back and forth along the $u-$axis, although both x and y are changing, their ratio $y(t)/x(t) = r_u$ remains fixed. This is what a normal mode is: a fixed relationship between the displacements of the individual oscillations (x and y). It is a pattern of displacements of the individual oscillations, which can be specified by their ratios. When only one mode is "*active*" (moving), all of the individual oscillators (x and y in this case) move in phase or in antiphase with one another at the frequence of that mode, but each has a different amplitude. Because they are in phase or antiphase, all cross the equilibrium position at the same time and, when the displacement of one is a maximum, the displacements of all the others are either maxima or minima. In the example above, x and y are in phase when the oscillation is along the $u-$axis because $r_u > 0$, and they are in antiphase ($x > 0 \implies y < 0$, and vice-versa) when it is along the $v-$axis ($r_v < 0$).

However, the point of this chapter is to show how motion involving two or more degrees of freedom can actually be described in a very simple way. The motion of two coupled oscillators can be modelled as a **superposition** of *independent* motion of two uncoupled simple harmonic oscillators - each with its own phase and amplitude. There is a catch: the degrees of freedom that oscillate independently of one another are not x and y, but linear combinations of x and y. A linear combination of x and y is a **collective coordinate** because it simultaneously gives information about both x and y. If we are given the values of two different collective coordinates, such as

$$u(t) = a_u\, x(t) + b_u\, y(t)$$
$$v(t) = a_v\, x(t) + b_v\, y(t)$$

we can always solve the simultaneous equations to find out what x and y are. If there are enough of them (as many as there are degrees of freedom), the values of a set of collective coordinates provides the same information that the values of the displacements of the oscillators provides. The collective coordinates can, themselves, be regarded as degrees of freedom. They are just a different choice of the set of degrees of freedom used to describe the dynamical system.

Every stable mechanical system at or near equilibrium has one special set of collective coordinates. These are the collective coordinates which oscillate independently of one another (x and y don't oscillate independently of one another) and are known as the **normal modes** of oscillation/vibration of the system. As we will see, the ratios a_u/b_u and a_v/b_v define the normal modes of vibration. The coordinates u and v are known as the **normal mode coordinates** and, because they are collective variables, changing u or v changes the positions of *all* of the coupled oscillators (x and y).

For the orange-in-bowl example, describing the dynamics of the system in terms of normal modes can be understood as a change of coordinate system. We can rotate the Cartesian axes so that they align with two perpendicular directions along which the motions are independent of one another, i.e., if $\tilde{V}(u, v)$ is the potential energy as a function of u and v and $\tilde{F}_u$ and $\tilde{F}_v$ are the components of the force in the new coordinate system, then

$$\tilde{F}_u(0, v) = -\left.\frac{\partial \tilde{V}}{\partial u}\right|_{(0,v)} = 0 \text{ and } \tilde{F}_v(u, 0) = -\left.\frac{\partial \tilde{V}}{\partial v}\right|_{(u,0)} = 0$$

This is illustrated in Figure 2.1. Before looking at the example of coupled pendulums in the next section, I want to explain a slight complication that only arises when coupled oscillators have different masses.

Coupled oscillators that have different masses

The mass of the orange obviously does not depend on the direction in which it is rolling. Therefore, when it is placed on axis u, the components of both the force and the acceleration along the perpendicular axis, v, are zero. When it is placed on axis v their components along axis u are both zero. Therefore, if it is placed somewhere along either the $u-$axis or the $v-$axis, and has zero initial velocity, it will perform a one dimensional oscillation along the axis on which it is placed. Later in this chapter we will represent the positions of two coupled oscillators as a point in a two dimensional plane. The situation will be exactly the same as for the orange rolling in the bowl when both oscillators have the same mass.

However, when they have different masses, they do not oscillate along the axes on which the force in the perpendicular direction is zero. They oscillate along axes on which the acceleration in the perpendicular direction is zero. To understand the difference, let us write the force on the orange as a vector, $\vec{F}$, and let us pretend that its mass depends on its direction, i.e.

$$\vec{F} = F_x\hat{x} + F_y\hat{y} = m_x\ddot{x}\hat{x} + m_y\ddot{y}\hat{y}$$

$\vec{F}$ is not parallel to the acceleration $\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$ unless $m_x = m_y$.

The two axes along which one dimensional oscillations can occur are those along which *accelerations* in the perpendicular direction are zero. In general, those two axes are *not* perpendicular to one another.

2.2 Coupled Pendulums

Let us consider the example of two ideal pendulums (massless inextensible strings), of length ℓ, which are attached to one another by a spring of force constant k, as shown in Figure 2.2*. Let us assume that the natural length of the spring is the same as the distance

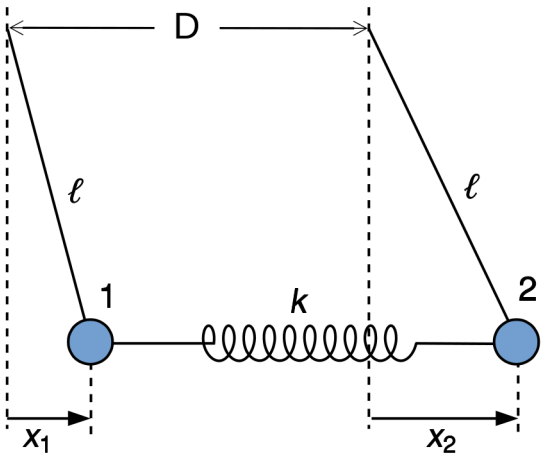


Figure 2.2:

between the points of support of the pendulums, D, so that, when both pendulums are hanging vertically, the system is at a stable equilibrium.

During motion, the horizontal components of the displacements from equilibrium of the left-hand and right-hand pendulums are x_1 and x_2, respectively. If we know $x_1(t)$ and $x_2(t)$ and their time derivatives, we have a complete description of the state of the pendulums.

We are going to make the simplifying assumption that $|x_1| \ll \ell$ and $|x_2| \ll \ell$ at all times. This assumption means that the angles that the pendulums make with the vertical are very

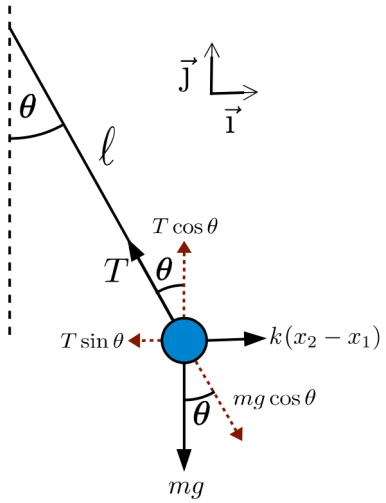
*I expect you to deduce and solve the equation of motion for a single pendulum on your own. If you run into problems, consult a textbook, Google it, or ask me during my office hours

small. We also assume that the natural length of the spring, D, is much greater than both $|x_1|$ and $|x_2|$. This is important because it means that the spring remains approximately horizontal throughout the motion and so the magnitude of its extension or compression is well approximated by $|x_2 - x_1|$.

2.2.1 Equations of motion

Clearly x_1 and x_2 are coupled by the spring. One way to find their coupled equations of motion is to find the net force on each mass and to apply Newton's second law. The net forces could be found by examining each mass independently to deduce all the forces acting on them, or by finding an expression for the potential energy $V(x_1, x_2)$ and taking partial derivatives. I'll use the first method.

Resolving forces



Each mass feels a vertical gravitational force

$$\vec{W} = -mg\,\vec{\jmath} \tag{2.1}$$

a tension force from the string

$$\vec{T} = -T\sin\theta\,\vec{\imath} + T\cos\theta\,\vec{\jmath} \tag{2.2}$$

and a force from the spring, $\vec{F}^{spring}$. The net force is the sum $\vec{F} = \vec{W} + \vec{T} + \vec{F}^{spring}$.

Small angle limit

We are interested in small oscillations. Therefore, we consider the limit $\theta \to 0$. In this limit

$$\sin\theta = \frac{x}{\ell} \approx \theta,$$
$$\cos\theta \approx 1,$$
$$\text{and} \qquad \vec{F}^{spring} \perp \vec{T}$$

where x is the *horizontal* displacement of the pendulum from equilibrium.

We can deduce the magnitude of $\vec{T}$ by noting that the mass always moves perpendicularly to the string. Therefore, its net force parallel to the string must be zero and so the component of the weight parallel to the string must exactly cancel $\vec{T}$.

$$T = mg\cos\theta \approx mg \tag{2.3}$$

The tension force can now be written as

$$\vec{T} \approx -mg\frac{x}{\ell}\,\vec{\imath} + mg\,\vec{\jmath} \tag{2.4}$$

If both x_2 and x_1 are positive, and if $x_2 > x_1$, the spring is extended. Therefore it will pull the left-hand mass to the right and the right hand mass to the left. The forces exerted by the spring on masses 1 and 2 are

$$\vec{F}_1^{\text{spring}} \approx k(x_2 - x_1)\,\vec{\imath} \tag{2.5}$$

and

$$\vec{F}_2^{\text{spring}} \approx -k(x_2 - x_1)\,\vec{\imath}, \tag{2.6}$$

respectively. When $|x_2| \to 0$ and $|x_1| \to 0$, the spring is horizontal and '$\approx$' becomes '='.

You should satisfy yourself that expressions 2.5 and 2.6 are always correct, both in magnitude and direction, regardless of the signs of x_1 and x_2.

The components of the net forces on masses 1 and 2 along $\vec{\imath}$ are $F_1 = \left(\vec{W} + \vec{T}_1 + \vec{F}_1^{\text{spring}}\right)\cdot\vec{\imath}$ and $F_2 = \left(\vec{W} + \vec{T}_2 + \vec{F}_2^{\text{spring}}\right)\cdot\vec{\imath}$. In the small angle limit these are

$$\text{Mass 1:} \qquad F_1 = \underbrace{-\frac{mg}{\ell}x_1}_{\text{pendulum weight}} + \underbrace{k(x_2 - x_1)}_{\text{coupling spring}}$$

$$\text{Mass 2:} \qquad F_2 = \underbrace{-\frac{mg}{\ell}x_2}_{\text{pendulum weight}} - \underbrace{k(x_2 - x_1)}_{\text{coupling spring}}$$

Applying Newton's second law, we get

$$\begin{aligned} m\ddot{x}_1 + \frac{mg}{\ell}x_1 - k(x_2 - x_1) &= 0 \\ m\ddot{x}_2 + \frac{mg}{\ell}x_2 + k(x_2 - x_1) &= 0 \end{aligned} \tag{2.7}$$

which we can rearrange to give

$$\begin{aligned} \ddot{x}_1 + \left(\frac{g}{\ell} + \frac{k}{m}\right)x_1 - \frac{k}{m}x_2 &= 0 \\ \ddot{x}_2 + \left(\frac{g}{\ell} + \frac{k}{m}\right)x_2 - \frac{k}{m}x_1 &= 0 \end{aligned} \tag{2.8}$$

It is sometimes useful to define

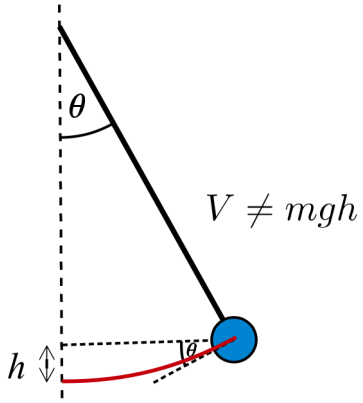
$$\omega_p \equiv \sqrt{\frac{g}{\ell}}; \qquad \omega_k \equiv \sqrt{\frac{k}{m}} \tag{2.9}$$

and to write the equations of motion as

$$\begin{aligned} \ddot{x}_1 + \omega_p^2 x_1 - \omega_k^2 (x_2 - x_1) &= 0 \\ \ddot{x}_2 + \omega_p^2 x_2 + \omega_k^2 (x_2 - x_1) &= 0 \end{aligned} \tag{2.10}$$

where ω_p is the "natural" angular frequency of the pendulums, i.e. what their angular frequency would be if they were not coupled, and ω_k is the natural angular frequency of a single mass attached to a fixed point by the spring.

Potential Energy



Expressed in terms of x_1 and x_2 the potential energy, in the small angle limit, is

$$V(x_1, x_2) = \frac{mg}{2\ell}\left(x_1^2 + x_2^2\right) + \frac{1}{2}k(x_2 - x_1)^2 \tag{2.11}$$

where the second term is the energy stored in the spring, and the first term is a gravitational contribution. The latter is not simply mg multiplied by the heights of the masses, because the string's tension counterbalances the weight to varying degrees as the angle changes. I'll leave it to you to derive this.

Using the definitions of ω_p and ω_k we can write

$$V(x_1, x_2) = \frac{1}{2}m\left(\omega_p^2 + \omega_k^2\right)\left(x_1^2 + x_2^2\right) - m\omega_k^2 x_1 x_2 \tag{2.12}$$

From the potential energy we can easily recover the equations of motion using

$$F_1 = -\frac{\partial V}{\partial x_1} = m\ddot{x}_1 \tag{2.13}$$

$$F_2 = -\frac{\partial V}{\partial x_2} = m\ddot{x}_2 \tag{2.14}$$

Coupled degrees of freedom

It is clear from Equations 2.10 and 2.12 that the degrees of freedom, x_1 and x_2, are coupled to one another. The horizontal acceleration of Mass 1 ($\ddot{x}_1$) is a function of *both* x_1 and x_2. Therefore, the acceleration of x_1 at any moment depends on the instantaneous value of x_2, and vice-versa.

However, we did not need to specify the state of the coupled pendulums using $x_1(t)$ and $x_2(t)$. We could instead have chosen to describe the state by the angles of the pendulums, $\theta_1(t)$ and $\theta_2(t)$, or by the values of $\alpha(t) \equiv 5x_1(t) + 17x_2(t)$ and $\beta(t) \equiv -32x_1(t) + x_2(t)/476$. In fact, there are an infinite number of ways to specify the state of the dynamical system. Some are more sensible than others (α and β are ludicrous choices).

Let's see what happens if we choose to specify the state of the system by the displacement of the midpoint between the masses from its equilibrium position

$$u_1(t) = \frac{x_1(t) + x_2(t)}{2} \tag{2.15}$$

and the change in the distance between the masses from its equilibrium value (D)

$$u_2(t) = x_2(t) - x_1(t) \tag{2.16}$$

To find the equations of motion for u_1 and u_2, we first write x_1 and x_2 in terms of them.

$$x_1 = u_1 - \frac{u_2}{2} \tag{2.17}$$

$$x_2 = u_1 + \frac{u_2}{2} \tag{2.18}$$

Now we substitute into Eq. 2.10 to give

$$\ddot{u}_1 - \frac{\ddot{u}_2}{2} + \omega_p^2 \left(u_1 - \frac{u_2}{2} \right) - \omega_k^2 u_2 = 0 \tag{2.19}$$

$$\ddot{u}_1 + \frac{\ddot{u}_2}{2} + \omega_p^2 \left(u_2 + \frac{u_2}{2} \right) + \omega_k^2 u_2 = 0 \tag{2.20}$$

By adding the two equations we get an equation of motion for u_1, and by subtracting one from the other we get an equation of motion for u_2.

$$\ddot{u}_1 + \omega_p^2 u_1 = 0 \tag{2.21}$$

$$\ddot{u}_2 + \left(\omega_p^2 + 2\omega_k^2 \right) u_2 = 0 \tag{2.22}$$

These equations are not coupled ($\ddot{u}_1$ does not depend on u_2 and $\ddot{u}_2$ does not depend on u_1). Therefore, the dynamics of u_1 and u_2 are independent of one another and both are simple harmonic - with frequencies $f_1 = 2\pi\omega_p$ and $f_2 = 2\pi\sqrt{\omega_p^2 + 2\omega_k^2}$, respectively. Because they are equations of simple harmonic oscillators, the solutions of Eqs. 2.21 and 2.22 have the form

$$u_1(t) = A_1 \cos\left(2\pi f_1 t + \phi_1\right) \tag{2.23}$$

$$u_2(t) = A_2 \cos\left(2\pi f_2 t + \phi_2\right) \tag{2.24}$$

where A_1, A_2, ϕ_1, and ϕ_2 are constants determined by the initial conditions.

Having found u_1 and u_2, we might want to find expressions for $x_1(t)$ and $x_2(t)$ so that we can track the motions of the individual pendulums. Combining Eqs. 2.17 and 2.18 with Eqs. 2.23 and 2.24 gives us the expressions

$$x_1(t) = A_1 \cos\left(\omega_1\, t + \phi_1\right) - \frac{1}{2} A_2 \cos\left(\omega_2\, t + \phi_2\right) \tag{2.25}$$

$$x_2(t) = A_1 \cos\left(\omega_1\, t + \phi_1\right) + \frac{1}{2} A_2 \cos\left(\omega_2\, t + \phi_2\right) \tag{2.26}$$

Therefore, the motion of each pendulum is a superposition of sinusoidal motions - one from each normal mode.

Note that, to find the normal modes, it is usually not as easy as adding and subtracting the original equations of motion. It happens to work in this case due to the left-right symmetry of the dynamical system. In Sec. 2.2.1 I will explain a more general approach to finding the normal modes, which doesn't require this symmetry.

Potential energy in normal mode coordinates

Combining Equations 2.12, 2.17, and 2.18 we can write the potential energy as a function of u_1 and u_2:

$$V(u_1, u_2) = m\omega_p^2 u_1^2 + \frac{1}{4} m(\omega_p^2 + 2\omega_k^2) u_2^2 \tag{2.27}$$

This is a quadratic function of both variables with no cross terms (no terms involving both u_1 and u_2). The importance of this is explained at the end of the Appendix to this chapter.

It might seem strange that a dynamical system with two coupled degrees of freedom can be re-expressed as two independent dynamical systems with one degree of freedom each, but try to visualize the motion described by $u_1(t)$ and $u_2(t)$. If only $u_2(t)$ is active (i.e. $A_1 = 0$) the displacement of one pendulum from the vertical position is always equal and opposite to the displacement of the other. Throughout the motion, there is perfect left-right symmetry and there is no reason why this symmetry would break (assuming the system was isolated). Breaking the left-right symmetry is the only way that the position of the midpoint could change to make $u_1(t)$ different from zero.

Now try to visualize the motion when only $u_1(t)$ is active ($A_2 = 0$). The position of the midpoint changes, but not the distance between the pendulums. They swing back and forth together and, at every instant, the distance of each pendulum from equilibrium would be identical ($x_1(t) = x_2(t)$). The spring would not be stretched, because $|x_1 - x_2| = 0$, and so neither pendulum would feel a force from the spring. Because the pendulums are identical there would be no reason for them to stop moving synchronously, and the distance between them could only change, thereby making u_2 differ from zero, if their synchronicity was broken.

If you think about it, there are very good symmetry reasons for these two degrees of freedom to move independently of one another.

What we have done to turn a coupled dynamical system (x_1 and x_2) into an apparently-uncoupled system (u_1 and u_2) is that we have made the coupling between the pendulums implicit in our choice of degrees of freedom. Previously, each degree of freedom, x_1 and x_2, referred to only one pendulum and so the coupling appeared in the equations of motion. Both u_1 and u_2 describe the *combined* system: each depends on the states of *both* pendulums and so they may be regarded as collective variables.

It turns out that u_1 and u_2 are a very special kind of collective variable known as **normal mode coordinates**.

2.3 Properties of normal modes

A normal mode multiplied by a real number is the same normal mode

Although multiplying of a normal coordinate by any real number changes its value, the rescaled coordinate represents the same mode of vibration. This can be understood by

looking at the equations of motion of, say, $u_2 = x_2 - x_1$ and $\tilde{u}_2 = 3x_1 - 3x_2 = -3u_2$.

$$\ddot{u}_2 = -\omega_2^2 u_2 \implies -3\ddot{u}_2 = 3\omega_2^2 u_2 \implies \ddot{\tilde{u}}_2 = -\omega_2^2 \ddot{\tilde{u}}_2$$

u_2 and $-3\,u_2$ have the same equation of motion. The reason is that the solution to one is equivalent to the solution to the other.

$$\begin{aligned} & u_2(t) = A_2 \cos(\omega_2 t + \phi_2) \\ \implies & -3u_2(t) = -3A_2 \cos(\omega_2 t + \phi_2) = 3A_2 \cos(\omega_2 t + \phi_2 + \pi) \\ \therefore\ & \tilde{u}_2(t) = \tilde{A}_2 \cos\left(\omega_2 t + \tilde{\phi}_2\right) \end{aligned}$$

Remember that the solution has a constant prefactor (A_2 or $\tilde{A}_2$) and a constant in its phase (ϕ_2 or $\tilde{\phi}_2$) and these constants are determined by the initial conditions. The choices of u_2 and $\tilde{u}_2$ as the second normal mode are equivalent because the factor of three by which they differ can simply be absorbed into the definition of the amplitude and the change in sign can be absorbed into the definition of the phase constant. The key point is that, at any time t, the value of $x_2(t)/x_1(t)$ is the same for u_2 and $\tilde{u}_2 = -3\,u_2$.

For the coupled pendulums we could have chosen our normal mode variables to be

$$\begin{aligned} \tilde{u}_1(t) &= -12\,(x_1(t) + x_2(t)) = -24\,u_1(t) \\ \tilde{u}_2(t) &= \quad 47\,(x_2(t) - x_1(t)) = \quad 47\,u_2(t) \end{aligned}$$

and we would find that the equations of motion for $\tilde{u}_1$ and $\tilde{u}_2$ are not coupled and that their frequencies are the same as u_1 and u_2, respectively.

A mode of vibration is characterised by a pattern of displacements specified by the *ratios* of displacements from equilibrium of the individual oscillators. If this ratio is negative it means that the individual oscillators move in opposite directions and if it is positive it means that they move in the same direction. If we had N coupled oscillators, we could express the i^{th} normal mode as*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} = x_1(t) \begin{pmatrix} 1 \\ r_2^{(i)} \\ \vdots \\ r_N^{(i)} \end{pmatrix}$$

*I am expressing the state of the system of coupled oscillators as a vector. I could use many different notations for this, including

$$\begin{aligned} \vec{x} &= x_1\,\hat{\imath} + x_2\,\hat{\jmath} + \cdots \\ \vec{x} &= x_1\,\hat{x}_1 + x_2\,\hat{x}_2 + \cdots \\ \vec{x} &= (x_1, x_2, \cdots, x_N) \\ \vec{x} &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \end{aligned}$$

All of these are equivalent.

Uniqueness

Instead of u_1 and u_2 we could have chosen different collective variables. For example,

$$\alpha_1 = 3x_1 - x_2 \quad \text{and} \quad \alpha_2 = x_2 - 2x_1$$
$$\Rightarrow x_1 = \alpha_1 + \alpha_2 \quad \text{and} \quad x_2 = 2\alpha_1 + 3\alpha_2$$

α_1 and α_2 are collective variables in the sense that they refer to both pendulums and if we know them we have a complete knowledge of the state of the pendulums. However, if we were to write down equations of motion for α_1 and α_2 (try it) we would find that they are coupled to one another.

Normal modes are unique, up to a multiplicative constant. Any choices of collective variables *except* multiples of u_1 and u_2 would have coupled equations of motion.

Two common choices of the arbitrary multiplicative factor in a mode's definition

Choice 1:
It is common to choose the normal coordinates such that they are **normalised**[*]. By this I mean that if, for example, the first normal mode coordinate is $u_1 = a_1 x_1 + b_1 x_2$ or any multiple of it, the choice

$$u_1 = a_1' x_1 + b_1' x_2 = \frac{1}{\sqrt{a_1^2 + b_1^2}} (a_1 x_1 + b_1 x_2)$$

is often made. This is because $\sqrt{a_1'^2 + b_1'^2} = 1$. Sometimes this can simplify the mathematics or clarify the interpretation of some of the constants, as we will see, but it is really just a matter of preference. By normalising the normal mode coordinates, we normalise the normal mode eigenvectors (defined below).

Choice 2:
Another common choice is

$$u_1 = \frac{1}{a_1} (a_1 x_1 + b_1 x_2) = x_1 + \frac{b_1}{a_1} x_2$$

This choice makes clear that what is important is not the prefactors a_1 and b_1, but their ratio and whether or not they have the same sign. I will sometimes define

$$r_2 \equiv -\frac{a_1}{b_1} \quad \text{and} \quad r_1 \equiv -\frac{a_2}{b_2}$$
$$\Longrightarrow \quad u_1 = x_1 - \frac{1}{r_2} x_2 \quad \text{and} \quad u_2 = x_1 - \frac{1}{r_1} x_2$$

The reason for this notation becomes clear if we consider the motion when only one mode is active. If only Mode 1 is active, then $u_2 = 0 \implies x_2 = r_1 x_2$ and so Mode 1 is a motion along the line of slope r_1 passing through the origin in two dimensional $x_1 x_2$ Cartesian plane. If only Mode 2 is active, then $u_1 = 0 \implies x_2 = r_2 x_2$ and so Mode 2 is a motion along the line of slope r_2 passing through the origin.

[*]Usually, the term "normalised" refers to a vector. We normalise a vector $\vec{v} = v_1 \hat{x}_1 + v_2 \hat{x}_2$ by multiplying it by a number r such that the new definition of $\vec{v}$ ($= r \times$ old $\vec{v}$)) has a magnitude of one. This number is $r = 1/(v_1^2 + v_2^2)$.

If only one normal mode is active, all oscillators move in phase or in antiphase

Consider the following normal mode coordinates of a pair of coupled oscillators

$$\begin{aligned} u_1(t) &\equiv \quad 3x_1(t) + 4x_2(t) = A_1 \cos(\omega_1 t + \phi_1) \\ u_2(t) &\equiv -8x_1(t) + 6x_2(t) = A_2 \cos(\omega_2 t + \phi_2) \end{aligned} \tag{2.28}$$

Now suppose that $u_2(t) = 0$ and $\dot{u}_2(t) = 0$ at some time t. Because the equations of motion of normal mode coordinates aren't coupled, this means that u_2 and $\dot{u}_2$ are zero at all times, i.e., $A_2 = 0$. Therefore,

$$\begin{aligned} u_2 = A_2 \cos(\omega_2 t + \phi_2) = -8x_1 + 6x_2 = 0 \implies x_2 = 4\,x_1/3 \text{ and } x_1 = 3\,x_2/4 \\ \therefore\, u_1(t) = 3\,x_1(t) + 4 \times \left(\frac{4}{3}\,x_1(t)\right) = A_1 \cos(\omega_1 t + \phi_1) \\ \implies x_1(t) = \frac{3}{25} A_1 \cos(\omega_1 t + \phi_1) \end{aligned}$$

and

$$\begin{aligned} u_1(t) = 3 \times \left(\frac{3}{4}\,x_2(t)\right) + 4\,x_2(t) = A_1 \cos(\omega_1 t + \phi_1) \\ \implies x_2(t) = \frac{4}{25} A_1 \cos(\omega_1 t + \phi_1) \end{aligned}$$

What this shows is that if only normal mode one is active ($\equiv$ moving/oscillating), x_1 and x_2 oscillate **in phase** with one another but with different amplitudes. Their amplitudes are in the ratio $3 : 4$.

If we assume that only mode two is active ($\implies A_1 = 0$) we find, by repeating the procedure above, that

$$\begin{aligned} x_1(t) &= -\frac{4}{25} A_2 \cos(\omega_2 t + \phi_2) = \frac{4}{25} A_2 \cos(\omega_2 t + \phi_2 + \pi) \\ x_2(t) &= \quad \frac{3}{25} A_2 \cos(\omega_2 t + \phi_2) \end{aligned}$$

In this case the oscillators move **in antiphase**, meaning that they are always moving in opposite directions. The ratio of their amplitudes is $4 : 3$, in this case.

Normalized normal mode coordinates as displacement coordinates in a new vector basis

Representing coupled oscillators as a point in a Euclidean space:
We can specify the instantaneous state of a system of coupled oscillators as the position and velocity of a point on a two dimensional Cartesian plane. If the "x" and "y" coordinate axes are parallel to the unit vectors $\hat{x}_1$ and $\hat{x}_2$, respectively, where $|\hat{x}_1| = |\hat{x}_2| = 1$ and $\hat{x}_1 \cdot \hat{x}_2 \implies \hat{x}_1 \perp \hat{x}_2$, then the state $(x_1, \dot{x}_1, x_2, \dot{x}_2)$ of the coupled oscillators is completely specified by the point displaced from the origin by

$$\vec{x}(t) = x_1(t)\,\hat{x}_1 + x_2(t)\,\hat{x}_2$$

and by the velocity of that point,

$$\vec{v}(t) \equiv \dot{\vec{x}}(t) = \dot{x}_1(t)\,\hat{x}_1 + \dot{x}_2(t)\,\hat{x}_2.$$

If we know $\vec{x}(t)$ and $\vec{v}(t)$, we know x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$.

Specifying the same point using a different (rotated) set of axes:
Let's use the examples of normal mode coordinates specified in Eqs. 2.28 again, but let's normalize them (multiply them by a number that makes the sum of squares of the coefficients equal to one).

$$\begin{aligned} u_1(t) &= \frac{3}{5}\,x_1(t) + \frac{4}{5}\,x_2(t) \\ u_2(t) &= -\frac{4}{5}\,x_1(t) + \frac{3}{5}\,x_2(t) \end{aligned}$$

We found that, when only u_1 is active,

$$\frac{x_1}{x_2} = \frac{3}{4} \implies x_2 = \frac{4}{3}x_1$$

This means that when $A_2 = 0$, the point specified by $\vec{x}$ oscillates about the line through the origin of slope $4/3$ (recall: $y = mx$, where m = slope), which is equivalent to saying that it oscillates about the origin such that its displacement is always in the directions $\pm\,\hat{u}_1$, where

$$\hat{u}_1 = \frac{3}{5}\hat{x}_1 + \frac{4}{5}\hat{x}_2$$

When only normal mode two is active ($A_1 = 0$), the point oscillates about the line through the origin of slope $-3/4$, which means that its displacement is along $\pm\,\hat{u}_2$ where

$$\hat{u}_2 = -\frac{4}{5}\hat{x}_1 + \frac{3}{5}\hat{x}_2$$

These directions are perpendicular to one another

$$\hat{u}_1 \cdot \hat{u}_2 = \left(\frac{3}{5}\hat{x}_1 + \frac{4}{5}\hat{x}_2\right) \cdot \left(-\frac{4}{5}\hat{x}_1 + \frac{3}{5}\hat{x}_2\right) = -1 \tag{2.29}$$

and so we can express the displacement $\vec{x}$ as

$$\begin{aligned} \vec{x} &= \left(\vec{x}\cdot\hat{u}_1\right)\hat{u}_1 + \left(\vec{x}\cdot\hat{u}_2\right)\hat{u}_2 \\ &= \left(\frac{3}{5}\,x_1 + \frac{4}{5}\,x_2\right)\hat{u}_1 + \left(-\frac{4}{5}\,x_1 + \frac{3}{5}\,x_2\right)\hat{u}_2 \\ &= u_1\hat{u}_1 + u_2\hat{u}_2 \end{aligned}$$

Therefore, each normalized normal mode coordinate, u_i, is the displacement of the coupled system (represented by a point in the 2D Cartesian plane) along a particular direction, $\hat{u}_i$. The vectors $\hat{u}_i$, or any multiple of them, are known as the **normal mode eigenvectors** *. We can represent the *normalized* normal mode eigenvectors as

$$\hat{u}_1 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \quad \hat{u}_2 = \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$$

where

$$\hat{x}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{x}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

*You don't need to understand what I'm talking about in this footnote for now, but when, in years to come, you are fondly skimming back through these notes, you will know what a dynamical matrix is and you will also probably know that each $\hat{u}_i$ is an eigenvector of the dynamical matrix and that its eigenvalue is ω_i^2.

Orthogonality

In the examples discussed above, the normal mode eigenvectors are orthogonal (perpendicular) to one another. However, as discussed at the end of Sec. 2.1, this is only the case if the masses of the coupled oscillators are equal. When they are not, the eigenvectors are not orthogonal*. When masses are the same, motion along one eigenvector, say $\hat{u}_i$, which changes the normal mode coordinate u_i, does not change the value of any other normal mode coordinates, u_j, which would move it along $\hat{u}_j$. However, even when masses are not the same, motion along eigenvector $\hat{u}_i$ does not change the value of the acceleration, $\ddot{u}_j$, of any other normal mode coordinate, u_j, which would cause it to move along $\hat{u}_j$.

The eigenvectors and the normal modes can be thought of as completely independent ways (modes) in which the system can move. In the pendulum example, changing the distance between the pendulums by applying equal and opposite changes to x_1 and x_2 does not cause an acceleration of the position of the midpoint of the two pendulums. Similarly, changing the midpoint by applying equal, but not opposite, changes to x_1 and x_2, does not cause the second time derivative of the distance between the pendulums to change. A change in u_1 does not cause a change in u_2 and vice-versa.

N degrees of freedom $\Rightarrow$ N normal modes

There are always exactly as many distinct (i.e. not multiples) normal modes of a dynamical system as there are distinct degrees of freedom in the dynamical system. In the examples discussed so far, we started with two degrees of freedom (x_1 and x_2) and so we found two normal modes. The reason can be understood when we regard the normal mode coordinates as components of displacement in a new coordinate system.

The state of the system can be specified by a point, $\vec{x}(t)$, on a two dimensional plane because there are two degrees of freedom. All points in the plane can be written as a sum of displacements along only two perpendicular directions. There can't be a third eigenvector because any vector that is not a multiple of $\hat{u}_1$ or $\hat{u}_2$ can be written as a linear combination (the sum of multiples) of $\hat{u}_1$ and $\hat{u}_2$, and so is not orthogonal to either of them. If u_3 = constant $\times$ u_1 + constant $\times$ u_2, any change in u_3 changes both u_1 and u_2. Therefore, u_3 is not decoupled from u_1 or u_2.

Completeness

In two dimensions, any vector can be written in the form

$$\vec{v} = v_1\hat{x}_1 + v_2\hat{x}_2$$

as long as $\hat{x}_1$ and $\hat{x}_2$ are not parallel or antiparallel. Any vector can also be represented exactly as a linear combination of the two normal mode eigenvectors.

$$\vec{v} = \tilde{v}_1\hat{u}_1 + \tilde{v}_2\hat{u}_2$$

*They actually satisfy a more complicated orthogonality condition that involves a mass *matrix*, but this is beyond the scope of this course.

The set of basis vectors comprising $\hat{u}_1$ and $\hat{u}_2$ is **complete** in the sense that any vector which specifies the displacements (x_1, x_2) of the coupled system can be represented exactly as a linear combination of the normal mode eigenvectors.

The basis of normal mode eigenvectors is not the only complete basis. There are many others, but for any other choice, motion along one eigenvector causes acceleration along another.

2.4 Calculating and using the normal modes

We didn't really deduce the normal modes of the coupled pendulum in Sec. 2.2.1. We guessed them. It is often possible to guess normal modes because they are determined by the symmetry of the dynamical system. However, it is also possible to work out what they are mathematically. We'll do that in this section, with the help of an example.

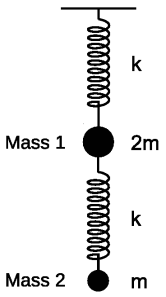


Figure 2.2: Example: Masses hanging by springs

Step 1: Find the equations of motion.

In the example above, x_1 is the vertical upward displacement from equilibrium of Mass 1 and x_2 is the vertical upward displacement from equilibrium of Mass 2. Satisfy yourself that the equations of motion are

$$\ddot{x_1} = -\omega_0^2 x_1 + \frac{1}{2}\omega_0^2 x_2 \tag{2.30}$$

$$\ddot{x_2} = \omega_0^2 (x_1 - x_2) \tag{2.31}$$

where $\omega_0^2 \equiv k/m$.

Step 2: Assume that only one mode is active and substitute a normal mode solution.

For two degrees of freedom (two dimensions), a normal mode oscillation occurs along a line $x_2 = rx_1$ in the Cartesian plane, where the x_1-axis is the $x-$axis and the x_2-axis is the $y-$axis. The normal mode coordinate is the displacement from the origin (the equilibrium) along this line, and it changes sinusoidally with some angular frequency ω. As we showed in Sec. 2.3, if only one mode is active, both x_1 and x_2 vary sinusoidally with angular frequency ω such that the relationship between them ($x_2 = r\,x_1$) remains fixed. Therefore, let us assume that only one mode is active, which implies that

$$\ddot{x}_1 = -\omega^2 x_1 \tag{2.32}$$

$$x_2 = r\,x_1 \Rightarrow \ddot{x}_2 = -\omega^2 r\,x_1 \tag{2.33}$$

and let us insert these relationships into the equations of motion to give us two equations in two unknown variables: the angular frequence, ω, and the ratio r that defines the normal

mode coordinate. Substituting into Eqs. 2.30 and 2.31 gives

$$-\omega^2 x_1 = -\omega_0^2 x_1 + \frac{1}{2}\omega_0^2 r x_1 \implies \omega^2 = \omega_0^2\left(1 - \frac{1}{2}r\right) \tag{2.34}$$

$$-\omega^2 r\, x_1 = \omega_0^2 x_1 (1 - r) \tag{2.35}$$

Step 3: Solve to find the normal coordinates and angular frequencies.

Eqs. 2.34 and 2.35 can be combined to give

$$-r\omega_0^2 + \frac{r^2}{2}\omega_0^2 = \omega_0^2 - r\omega_0^2 \implies r^2 = 2 \implies r = \pm\sqrt{2} \tag{2.36}$$

This means that the ratios and the angular frequencies for the two normal modes are (using 2.34)

$$r_1 = \quad \sqrt{2}, \quad \omega_1^2 = \omega_0^2\left(1 - \frac{1}{\sqrt{2}}\right) \tag{2.37}$$

$$\text{and } r_2 = -\sqrt{2}, \quad \omega_2^2 = \omega_0^2\left(1 + \frac{1}{\sqrt{2}}\right) \tag{2.38}$$

We can now write down the normal mode coordinates.

$$u_1(t) \propto x_1(t) - \frac{1}{r_2}x_2(t) = x_1(t) + \frac{1}{\sqrt{2}}x_2(t) \tag{2.39}$$

$$u_2(t) \propto x_1(t) - \frac{1}{r_1}x_2(t) = x_1(t) - \frac{1}{\sqrt{2}}x_2(t) \tag{2.40}$$

The reason for writing $u_1 \propto x_1 - x_2/r_2$ and a similar expression for u_2 is that r_2, which is associated with angular frequency ω_2, is the ratio x_2/x_1 when $u_1 = 0$. Similarly, $x_2/x_1 = r_1$ when $u_2 = 0$. The coordinates u_1 and u_2 are only determined up to a constant and so we can choose to multiply them by any convenient factors and they will still be independent of one another.

Step 4: Express $x_1(t)$ and $x_2(t)$ in terms of the normal modes.

Equations 2.39 and 2.40 can be used to express x_1 and x_2 in terms of u_1 and u_2. We find

$$x_1(t) = \frac{1}{2}u_1(t) + \frac{1}{2}u_2(t) \tag{2.41}$$

$$x_2(t) = \frac{1}{\sqrt{2}}u_1(t) - \frac{1}{\sqrt{2}}u_2(t) \tag{2.42}$$

and since we know that

$$u_1(t) = A_1\cos(\omega_1 t + \phi_1)$$
$$u_2(t) = A_2\cos(\omega_2 t + \phi_2)$$

where A_1, A_2, ϕ_1, ϕ_2 are determined by initial conditions, this can also be written as

$$x_1(t) = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) + \frac{A_2}{2}\cos(\omega_2 t + \phi_2)$$
$$x_2(t) = \frac{A_1}{\sqrt{2}}\cos(\omega_1 t + \phi_1) - \frac{A_2}{\sqrt{2}}\cos(\omega_2 t + \phi_2)$$

So the motion of each of the two masses is a superposition of two sinusoidal motions of different frequencies, corresponding to the two normal modes. To make this look a bit cleaner, we are free to define new amplitudes $B_1 = A_1/2$ and $B_2 = A_2/2$.

$$x_1(t) = B_1 \cos(\omega_1 t + \phi_1) + B_2 \cos(\omega_2 t + \phi_2) \tag{2.43}$$

$$x_2(t) = B_1\sqrt{2} \cos(\omega_1 t + \phi_1) - B_2\sqrt{2} \cos(\omega_2 t + \phi_2) \tag{2.44}$$

$$= r_1 B_1 \cos(\omega_1 t + \phi_1) + r_2 B_2 \cos(\omega_2 t + \phi_2) \tag{2.45}$$

2.5 Beating

Now I want to use the coupled oscillator to illustrate a general phenomenon known as **beating**. It occurs when two waves or oscillations of almost the same frequency are superimposed. For example, when two musical notes that have almost the same pitch are played simultaneously the volume of the sound can be heard to rise and fall repeatedly. This is because each note, which is a sound wave, causes your eardrum to oscillate. The **principle of superposition** tells us that the effect on your eardrum of the two note together is the sum of the effects of each note independently. Therefore, the motion of your eardrum can be described as the sum of two oscillations of different frequency.

We saw in simulations in lectures that, when two oscillators are coupled, the superposition of their motions appeared to be an oscillation with a time-varying amplitude. The amplitude repeatedly increased to a maximum and decreased to almost zero again. This section explains the reason for that behaviour.

For the coupled pendulums we found that the horizontal displacements of the masses from equilibrium could be written as sums of two normal mode displacements. That is

$$x_1(t) = u_1(t) - \frac{1}{2}u_2(t) = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \tag{2.46}$$

$$x_2(t) = u_2(t) + \frac{1}{2}u_2(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \tag{2.47}$$

where $\omega_1 = \omega_p$, $\omega_2 = \sqrt{\omega_p^2 + 2\omega_k^2}$ and A_1, A_2, ϕ_1, and ϕ_2, depend on initial conditions.

We say that x_1 is a **superposition** of two harmonic oscillations.

Let's consider what happens when the two frequencies are almost the same. This would be the case if $\omega_k \ll \omega_p$, which implies that the two pendulums are very weakly coupled, i.e. $\frac{k}{m} \ll \frac{g}{l}$. To simplify the mathematics let's assume that $A_1 = A_2 = A$ and $\phi_1 = \phi_2 = 0$. Let us consider the complex variable

$$\tilde{x}(t) = Ae^{i\omega_1 t} + Ae^{i\omega_2 t}$$

The real part of $\tilde{x}$ is x_2. If we define

$$\omega \equiv \frac{\omega_1 + \omega_2}{2} \quad \text{and} \quad \Delta\omega \equiv \frac{\omega_2 - \omega_1}{2}$$

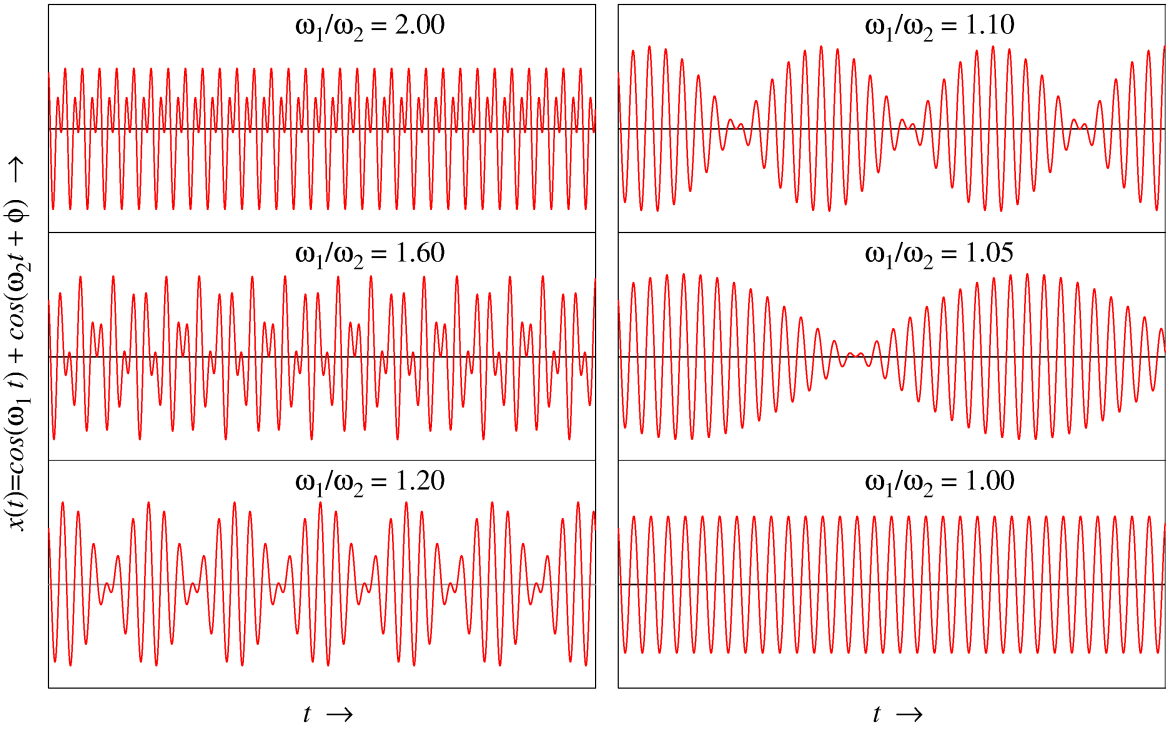


Figure 2.3: The sum of two sinusoids of equal amplitude and angular frequencies ω_1 and ω_2, for various values of ω_1/ω_2. When $\omega_1/\omega_2 = 1$, the sum is simply a sinusoid with double the amplitude. When $\omega_1/\omega_2 \neq 1$, the sum can be a more complicated oscillation. As ω_1/ω_2 approaches 1 an interesting pattern appears. The sum of the sinusoids looks like a high-frequency sinusoid with an amplitude that is varying sinusoidally at a much lower frequency. This phenomenon is known as "beating".

then

$$\begin{aligned} \omega_1 &= \omega - \Delta\omega, \qquad \omega_2 = \omega + \Delta\omega \\ \Rightarrow \tilde{x}(t) &= Ae^{i(\omega-\Delta\omega)t} + Ae^{i(\omega+\Delta\omega)t} \\ &= Ae^{i\omega t}\left(e^{i\Delta\omega t} + e^{-i\Delta\omega t}\right) \\ &= 2Ae^{i\omega t}\cos(\Delta\omega t) \\ \therefore x_2(t) &= 2A\cos(\omega t)\cos(\Delta\omega t) \end{aligned} \tag{2.48}$$

In a similar manner we could show that

$$x_1(t) = 2A\cos(\omega t)\sin(\Delta\omega t) \tag{2.49}$$

Since $\Delta\omega \ll \omega$, x_1 and x_2 have the form of fast oscillations modulated by a slow oscillation, as illustrated in Figure 2.4. The period of the fast oscillation is $T_{fast} = \dfrac{2\pi}{\omega}$ and the period of the slow modulation is $T_{slow} = \dfrac{2\pi}{\Delta\omega}$. However, be careful: *half* of T_{slow} is known as the **beat period**.

$$T_{beat} \equiv \frac{T_{slow}}{2} \tag{2.50}$$

This is because, when you hear musical notes beating, your ear isn't sensitive to the phases of the waves. It is only sensitive to their amplitude and frequency. You hear a train of sound pulses which arrive at intervals of T_{beat}, as indicated in Figure 2.4.

If the two waves have frequencies $f_1 = \omega_1/2\pi$ and $f_2 = \omega_2/2\pi$, the **beat frequency** is

$$f_{\text{beat}} = |f_1 - f_2| \tag{2.51}$$

$$T_{\text{beat}} = \frac{1}{f_{\text{beat}}} = \frac{1}{|f_1 - f_2|} \tag{2.52}$$

As illustrated in Figure 2.5, the closer the frequencies get to one another, the longer the interval between the beats.

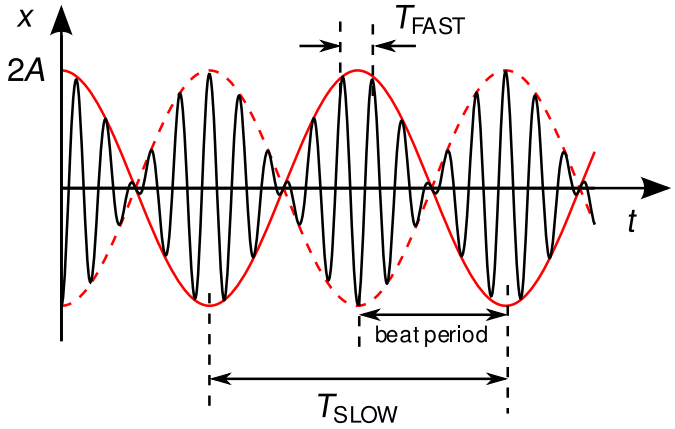


Figure 2.4:

2.5.1 Coupled pendulums

From Equations 2.17, 2.18, 2.23, and 2.24 it is clear that the horizontal displacement of each pendulum is a sum of two sinusoidally-varying contributions (u_1 and $\pm\frac{u_2}{2}$), which have frequencies

$$f_1 = 2\pi\sqrt{\frac{g}{\ell}} \tag{2.53}$$

$$f_2 = 2\pi\sqrt{\frac{g}{\ell} + 2\frac{k}{m}} \tag{2.54}$$

Therefore, if the two oscillations have close to the same frequency x_1 and x_2 will have the form of sinusoids whose amplitudes are modulated at a lower frequency. They will have close to the same frequency if

$$\frac{g}{l} \gg \frac{k}{m} \tag{2.55}$$

So, when two simple harmonic oscillators are weakly coupled, their displacements behave as in Figure 2.5. Energy passes from one pendulum to the other and back again. When the energy of one (kinetic+potential) is a maximum, the energy of the other is a minimum.

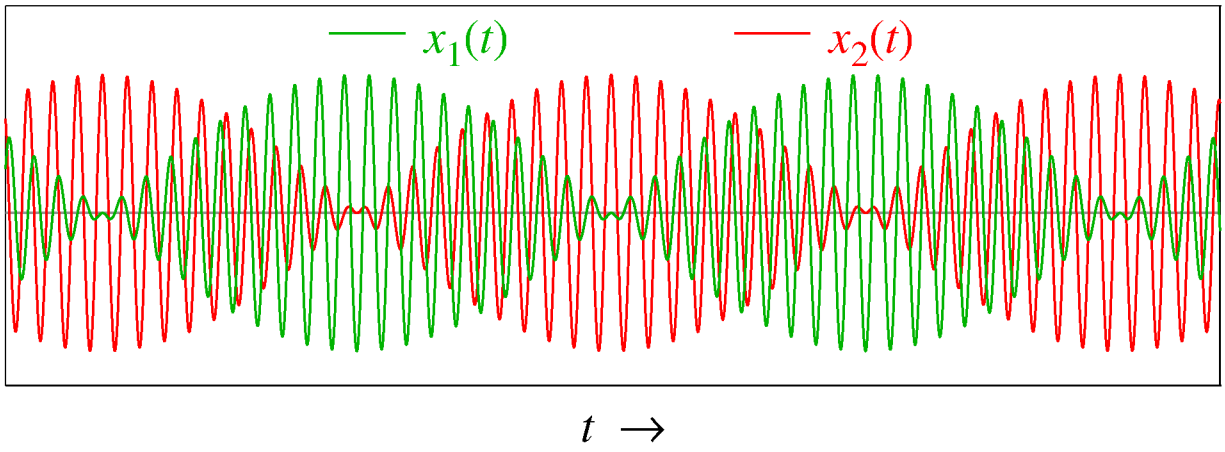


Figure 2.5: Horizontal displacements of the coupled pendulums as functions of time when they are weakly coupled ($\frac{k}{m} \ll \frac{g}{l}$).

Appendix: Potential energy of two coupled oscillators

In this appendix some of the general properties of the potential energy of any two coupled oscillations (two oscillators or an oscillator in two dimensions) are reviewed. Suppose that x and y are the displacements from equilibrium of two masses, m_x and m_y, respectively, and that their potential energy, relative to their potential energy at equilibrium, is $V(x, y)$. A Taylor expansion of $V(x, y)$ about the equilibrium at $(0, 0)$ is

$$V(x,y) = \cancel{V(0,0)}^{\,0} + \cancel{\frac{\partial V}{\partial x}\Big|_{(0,0)}}^{\,0} x + \cancel{\frac{\partial V}{\partial y}\Big|_{(0,0)}}^{\,0} y + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\Big|_{(0,0)} x^2 + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\Big|_{(0,0)} y^2 + \frac{1}{2}\frac{\partial^2 V}{\partial x \partial y}\Big|_{(0,0)} xy + \cdots \tag{2.56}$$

If we assume that x and y are both very small so that cubic and higher order terms can be neglected, and if we define

$$k_{xx} \equiv \frac{\partial^2 V}{\partial x^2}\Big|_{(0,0)}; \qquad \omega_{xx} \equiv \sqrt{\frac{k_{xx}}{m_x}}$$

$$k_{yy} \equiv \frac{\partial^2 V}{\partial y^2}\Big|_{(0,0)}; \qquad \omega_{yy} \equiv \sqrt{\frac{k_{yy}}{m_y}} \tag{2.57}$$

$$k_{yx} \equiv \frac{\partial^2 V}{\partial y \partial x}\Big|_{(0,0)} = k_{xy} \equiv \frac{\partial^2 V}{\partial x \partial y}\Big|_{(0,0)}; \qquad \omega_{xy} \equiv \sqrt{\frac{k_{xy}}{m_x}}, \quad \omega_{yx} \equiv \sqrt{\frac{k_{xy}}{m_y}} \tag{2.58}$$

we get

$$V(x,y) = \frac{1}{2}k_{xx}x^2 + \frac{1}{2}k_{yy}y^2 + k_{xy}x\,y \tag{2.59}$$

from which we can find the forces

$$\Rightarrow F_x = -\frac{\partial V}{\partial x} = m_x\ddot{x} = -k_{xx}x - k_{xy}y$$

$$\Rightarrow F_y = -\frac{\partial V}{\partial y} = m_y\ddot{y} = -k_{yy}y - k_{xy}x \tag{2.60}$$

giving us the two coupled equations of motion

$$\begin{aligned} \ddot{x} + \omega_{xx}^2 x + \omega_{xy}^2 y &= 0 \\ \ddot{y} + \omega_{yy}^2 y + \omega_{yx}^2 x &= 0 \end{aligned} \tag{2.61}$$

Both x and y are time-dependent variables. However, if $\omega_{xy} \neq 0$ or $\omega_{yx} \neq 0$, they can't be described independently because the acceleration of x at time t depends on $y(t)$ and vice-versa. This is the general form for two coupled equations of motion when the displacement from equilibrium is small enough that the potential is harmonic. They have the same form as Eqs. 2.10.

Returning briefly to Eq. 2.56 it is worth noting that if we wrote the potential energy as a function of the normal mode variables (u_1, u_2) and Taylor expanded about the equilibrium $(u_1, u_2) = (0, 0)$, we would have found

$$V(u_1, u_2) = \frac{1}{2} k_{11} u_1^2 + \frac{1}{2} k_{22} u_2^2 \tag{2.62}$$

$$\text{where} \qquad k_{11} \equiv \left. \frac{\partial^2 V}{\partial u_1^2} \right|_{(0,0)} = 2m\omega_p^2 \tag{2.63}$$

$$\text{and} \qquad k_{22} \equiv \left. \frac{\partial^2 V}{\partial u_2^2} \right|_{(0,0)} = \frac{1}{2} m \left(\omega_p^2 + 2\omega_k^2 \right) \tag{2.64}$$

It useful to compare this to Eq. 2.59 because it shows that if the potential energy, expressed in terms of the normal modes, is Taylor expanded to second order about its minimum, the cross derivative vanishes, i.e.,

$$k_{12} \equiv \left. \frac{\partial^2 V}{\partial u_1 \partial u_2} \right|_{(0,0)} = 0 \tag{2.65}$$

Chapter 3

Damped oscillations

3.1 Introduction

In previous lectures we have considered simple harmonic motion of a single oscillator and the collective motion of coupled oscillators. I now want to discuss energy dissipation.

As we know, in an isolated physical system the energy remains constant. As we also know, the energy never remains constant in practice. Or, at least, it doesn't remain in the form that we want it to. Isolated mechanical systems slow down and stop. Always.

There is always some friction or air resistance and so mechanical energy (kinetic and potential) turns to heat, sound, light, etc. Therefore, real oscillators slow down. Their motion is damped.

There are many kinds of harmonic oscillators, and therefore many different ways in which energy can be dissipated. For example, if the dominant dissipation mechanism is *Coulomb friction*, the friction force that resists the motion is independent of the speed $|\vec{v}|$, but not the velocity, $\vec{v}$, because it always acts in the direction $-\vec{v}/|\vec{v}|$. When the dominant dissipation mechanism is resistance from the fluid (usually air) through which it moves, the rate at which mechanical energy dissipates usually does depend on the speed. For a particle moving through a fluid, such as air or water, the force with which the fluid resists the particle's motion can often be well approximated as

$$\vec{F}_d = -b_1\vec{v} - b_2|\vec{v}|\vec{v} \tag{3.1}$$

Note that the **damping force**, $\vec{F}_d$, always points in the opposite direction to the velocity. It never speeds the oscillator up. We will consider the case

$$b_1|\vec{v}| \gg b_2|\vec{v}|^2 \tag{3.2}$$

and so we'll neglect the second term. However, you should be aware that there are many important situations and systems in which the second term dominates.

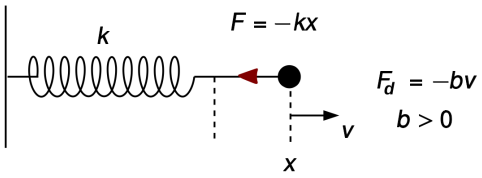


Figure 3.1:

3.2 Equation of motion of the damped harmonic oscillator

The net force on the damped oscillator is the sum of the force exerted by the spring and the damping force. Therefore

$$F = m\ddot{x} = -kx - bv \tag{3.3}$$

$$\Rightarrow m\ddot{x} + b\dot{x} + kx = 0 \tag{3.4}$$

where k is the spring constant and b is the **damping coefficient**. We can write this as

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{3.5}$$

where $\gamma \equiv \frac{b}{m}$ is the **damping constant**[*] and $\omega_0^2 \equiv \frac{k}{m}$ is the natural frequency of the oscillator in the absence of dissipation. Note that γ and ω_0 both have the dimensions of $1/\text{time}$ and their ratio $Q = \omega_0/\gamma$ is dimensionless.

I will solve an analogous equation to Equation 3.5 in the complex plane by trialling a solution of the form

$$\tilde{x}(t) = \tilde{A}e^{\alpha t} \tag{3.6}$$

where $\tilde{x}$, $\tilde{A} = \left|\tilde{A}\right|e^{i\phi}$, and α may all be complex variables/constants.

$$\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = 0 \tag{3.7}$$

$$\Rightarrow \alpha^2\tilde{x} + \alpha\gamma\tilde{x} + \omega_0^2\tilde{x} = 0$$

$$\therefore \alpha^2 + \alpha\gamma + \omega_0^2 = 0 \tag{3.8}$$

Note that, because the coefficients γ and ω_0^2 are real numbers, if we take the complex conjugate of Eq. 3.7 we find

$$\ddot{\tilde{x}}^* + \gamma\dot{\tilde{x}}^* + \omega_0^2\tilde{x}^* = 0$$

and so, if $\tilde{x}$ is a solution, so is its complex conjugate. Solving Eq. 3.8 for α gives

$$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{3.9}$$

[*]There is no consensus in the literature on either the terminology or the definition of γ. In some books γ is defined as $b/2m$. Sometimes γ is referred to as the *damping coefficient* or the *damping ratio*, and sometimes b is referred to as the *damping constant*. This is annoying and confusing, but it does not change the theory in any important way. What is important is consistency between the definition of γ used in the equation of motion, and later definitions and formulae (Q-factor, decay times, etc.).

We will now consider three different types of motion which can arise depending on the values of γ and ω_0. That is, depending on the strength of the damping and the natural angular frequency of the oscillator. If the damping is weak, it will change the motion very little over a period of the motion. In this case, the motion is still oscillatory. If the damping is strong enough, it will dissipate all the energy in less than a period of the motion. In this case, the motion is not oscillatory because there isn't time for one complete oscillation before the motion stops.

3.3 Light damping $\left(\gamma^2/4 < \omega_0^2\right)$

When γ is small enough that $\gamma^2/4 - \omega_0^2 < 0$, then α is a complex number

$$\alpha = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{3.10}$$

and we get the oscillatory solutions

$$\tilde{x}(t) = \tilde{A}e^{-\frac{\gamma}{2}t}e^{\pm i\omega_d t} = \left|\tilde{A}\right|e^{-\frac{\gamma}{2}t}e^{\pm i(\omega_d t+\phi)} \tag{3.11}$$

$$\text{where} \quad \omega_d^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{3.12}$$

ω_d is the **damped angular frequency**, which is independent of the amplitude. Note that damping reduces the frequency ($\omega_d < \omega_0$), which may roughly be understood as being due to the damping slowing down the motion so that each complete oscillation takes longer. The general solution is a linear combination of the $e^{i(\omega_d t+\phi)}$ and $e^{-i(\omega_d t+\phi)}$ solutions, i.e.

$$\tilde{x}(t) = e^{-\frac{\gamma}{2}t}\left[A_1 e^{i(\omega_d t+\phi)} + A_2 e^{-i(\omega_d t+\phi)}\right]$$

and to get the physical solution we take the real part of $\tilde{x}$, which is

$$x(t) = A_0 e^{-\frac{\gamma}{2}t}\cos\left(\omega_d t + \phi\right) \tag{3.13}$$

where the constants $A_0 = (A_1 + A_2)$ and ϕ are determined by initial conditions.

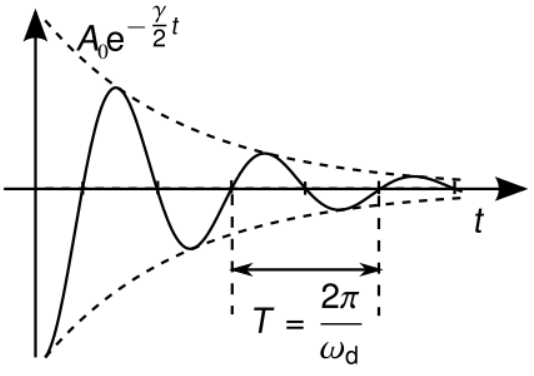


Figure 3.2:

If the damping is sufficiently weak, we can think of this as our normal sinusoidal solution, but with an exponentially decaying amplitude $A(t) = A_0 e^{-\frac{\gamma}{2}t}$, where the amplitude at $t = 0$ is A_0. If the damping is strong enough that $A(t)$ changes a lot during each complete oscillation, this picure is less useful.

The decay constant for the amplitude is $\gamma/2$, which means that the amplitude reduces by a factor $1/e$ in time $\tau_A = 2/\gamma$. Again, if the damping is weak enough that its amplitude is approximately constant over a period, the energy of the oscillator is well approximated by

$$E(t) \approx \frac{1}{2} m\omega_d^2 A(t)^2 = \frac{1}{2} m\omega_d^2 A_0^2 e^{-\gamma t} \tag{3.14}$$

and so it reduces by a factor $1/e$ in time $\tau_E \sim 1/\gamma$.

3.4 Q-factor

A quantity commonly used to characterise real oscillators is the **Q−factor**, which is dimensionless and defined as

$$Q \equiv \frac{\omega_0}{\gamma} \tag{3.15}$$

We can express ω_d in terms of ω_0 and Q as follows.

$$\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \omega_0 \sqrt{1 - \frac{\gamma^2}{4\omega_0^2}} \tag{3.16}$$

$$= \omega_0 \sqrt{1 - \frac{1}{4Q^2}} = \omega_0 \left[1 - \frac{1}{8Q^2} - \cdots \right] \tag{3.17}$$

We can see from this that the difference between ω_d and ω_0 decreases rapidly as Q increases. For example,

$$\begin{aligned}
Q = 0.5 &\Rightarrow \omega_d = 0.0\ \omega_0 \\
Q = 0.75 &\Rightarrow \omega_d \approx 0.745\ \omega_0 \\
Q = 1 &\Rightarrow \omega_d \approx 0.866\ \omega_0 \\
Q = 2 &\Rightarrow \omega_d \approx 0.968\ \omega_0 \\
Q = 10 &\Rightarrow \omega_d \approx 0.999\ \omega_0
\end{aligned}$$

Therefore, $\omega_d \approx \omega_0$ for most oscillators, and we can write

$$Q = \frac{\omega_0}{\gamma} \approx \frac{\omega_d}{\gamma} = \frac{2\pi/T}{2/\tau_A} = \pi \frac{\tau_A}{T} = \pi N \tag{3.18}$$

where N is the number of complete oscillations that occur before the amplitude of the oscillator has reduced by $1/e$.

The larger the $Q-$factor the lower the energy dissipation and the closer the frequency is to that of the undamped oscillator.

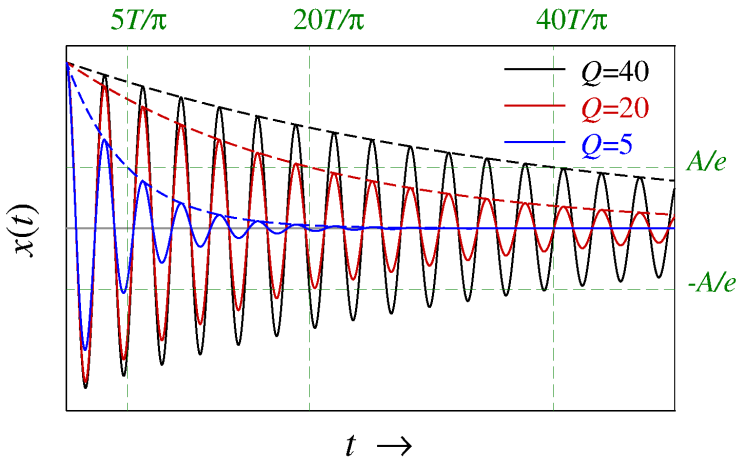


Figure 3.3:

3.5 Energy dissipation

The total energy of the oscillator is

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{3.19}$$

where

$$x(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi) \tag{3.20}$$

$$\Rightarrow \dot{x}(t) = -\frac{\gamma}{2}A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi) - \omega_d A_0 e^{-\frac{\gamma}{2}t}\sin(\omega_d t + \phi) \tag{3.21}$$

By substituting into Equation 3.19, you should satisfy yourself that, for large Q,

$$E(t) \approx \frac{1}{2}kA(t)^2 \tag{3.22}$$

$$\Rightarrow \frac{dE}{dt} = -\gamma E \tag{3.23}$$

This means that the change in energy in one period is approximately

$$\Delta E = \frac{dE}{dt}\frac{2\pi}{\omega_d} = -2\pi E\frac{\gamma}{\omega_d} \approx -\frac{2\pi}{Q}E \tag{3.24}$$

$$\Rightarrow \frac{Q}{2\pi} \approx \frac{E}{-\Delta E} = \frac{\text{energy stored in oscillator}}{\text{energy lost per period}} \tag{3.25}$$

3.6 Heavy damping $\left(\gamma^2/4 > \omega_0^2\right)$

When γ is large enough, such that $\frac{\gamma^2}{4} > \omega_0^2 \Rightarrow Q < \frac{1}{2}$ then $\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$ is a real number. This means that there is no oscillatory motion (which always requires a *complex* exponential) and a general solution to the equation of motion is

$$x(t) = e^{-\frac{\gamma}{2}t}\left[A_1 e^{t\sqrt{\frac{\gamma^2}{4}-\omega_0^2}} + A_2 e^{-t\sqrt{\frac{\gamma^2}{4}-\omega_0^2}}\right] \tag{3.26}$$

We can rewrite this in the form

$$x(t) = e^{-\frac{\gamma}{2}t}\left[B_1 \cosh\left(t\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right) + B_2 \sinh\left(t\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)\right] \tag{3.27}$$

This form is useful if one of the initial conditions is $x(0) = 0$ [$\dot{x}(0) = 0$] because then $B_1 = 0$ [$B_2 = 0$] and the cosh [sinh] term disappears.

3.7 Critical damping $\left(\gamma^2/4 = \omega_0^2\right)$

When $Q = \frac{1}{2} \Rightarrow \frac{\gamma^2}{4} = \omega_0^2$ then $\alpha = -\gamma/2$. If we return to our trial solution (Equation 3.6) we find

$$x(t) = A_0 e^{-\frac{\gamma}{2}t} \tag{3.28}$$

which can't be the general solution because it only contains one adjustable parameter. Solutions of second order differential equations have two constants of integration. Without proof, here is the general solution for the critically damped oscillator

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t} \tag{3.29}$$

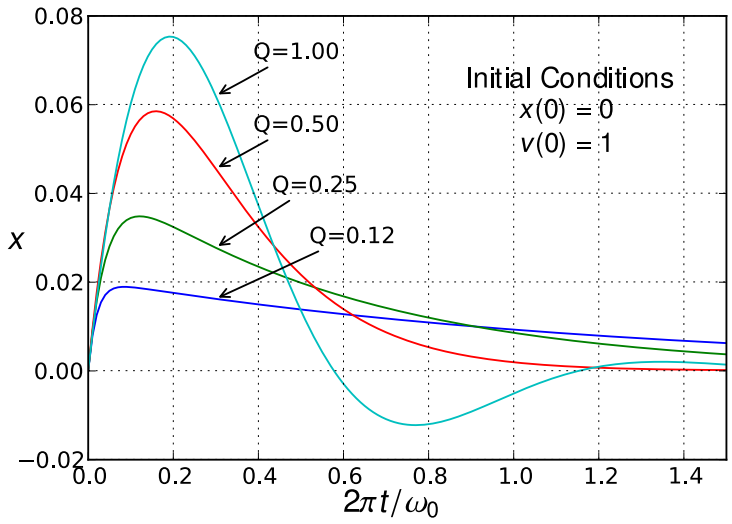


Figure 3.4: Plots of displacement as a function of time for oscillators with different $Q-$factors and initial conditions corresponding to an impulse ($x = 0$, $\dot{x} \neq 0$). The critically damped oscillator decays to zero the fastest.

Physical picture of light, heavy, and critical damping

To have pictures in your minds of the different kinds of damping, consider again the orange oscillating in the bowl, or a ball bearing rolling in a bowl. The ball slows down and gradually

comes to rest at the bottom of the bowl while oscillating. This is light damping. If the ball and bowl were immersed in water, the damping of the motion would be much heavierThe ball would roll relatively slowly to the bottom and its speed would be very low when it reached the minimum. It may overshoot the minimum, but probably only by a small amount and it would quickly come to rest. This system is very close to being critically damped. It barely oscillates, if at all. In a more viscous fluid the ball would not oscillate at all. It would simply roll very very slowly to the bottom of the bowl. This is strong damping. Critical damping is the weakest damping for which the ball does not roll past the minimum. It can be thought of as the weak limit of strong damping. A critically-damped ball could be moved into the strong damping regime by changing to a very slightly shallower (lower curvature) bowl (reduce curvature $\implies$ reduce ω_0 $\implies$ $\gamma^2/4 > \omega_0^2$). It could be moved into the weak damping regime by using a bowl with a slightly larger curvature so that ω_0 is greater ($\implies$ $\omega_0^2 > \gamma^2/4$) and it overshoots the bottom of the bowl at least once before coming to rest.

Chapter 4

Driven Oscillations

4.1 Introduction

There are many examples of oscillations which are stimulated or driven by an externally applied force. One example is an orange in a bowl on a rocking boat. Another example is when infrared radiation passes through an ionic material: the radiation's time-varying electric field accelerates positive and negative ions in opposite directions, causing the ionic bonds to vibrate.

Oscillators are important components of many electronic devices. Since the energy of an oscillator always dissipates, any device that is required to oscillate for a long time (relative to the decay time τ_A) must be driven or it will cease to function.

In this chapter we'll consider the behaviour of the damped harmonic oscillator when it is subjected to a sinusoidal driving force, $F_D(t)$.

4.2 Forced damped oscillator

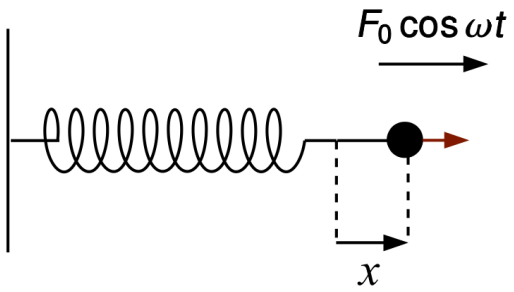


Figure 4.1:

Let's consider a mass (m) on a spring (k) driven by an applied force $F_D(t) = F_0 \cos \omega t$ and

with a damping force $-b\dot{x}$. The total force on the mass at any instant is

$$F = -kx - b\dot{x} + F_0 \cos \omega t \tag{4.1}$$

Applying Newton's second law ($F = m\ddot{x}$) gives the equation of motion

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t \tag{4.2}$$

which we can cast into the standard form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \tag{4.3}$$

where

$$\gamma \equiv \frac{b}{m}, \quad \omega_0^2 = \frac{k}{m} \tag{4.4}$$

Equation 4.3 is an **inhomogeneous** second order differential equation. Its associated **homogeneous** differential equation is the equation of motion of the damped oscillator (Equation 3.5).

Instead of solving Equation 4.3 for the real variable $x(t)$, we seek a solution in the complex plane of the equation

$$\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2 \tilde{x} = \frac{F_0}{m} e^{i\omega t} \tag{4.5}$$

which we solve by substituting the trial solution $\tilde{x}(t) = Ae^{i(\omega t+\phi)}$, to give the characteristic equation*

$$\left(-\omega^2 + i\omega\gamma + \omega_0^2\right) Ae^{i(\omega t+\phi)} = \frac{F_0}{m} e^{i\omega t} \tag{4.6}$$

After dividing across by $e^{i\omega t}$ and rearranging we get

$$Ae^{i\phi} = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\omega\gamma} \tag{4.7}$$

$$\implies A(\omega_0^2 - \omega^2) + iA\omega\gamma = \frac{F_0}{m} e^{-i\phi} = \frac{F_0}{m} \cos\phi + i\frac{F_0}{m} \sin\phi \tag{4.8}$$

$$\therefore \cos\phi = \frac{m(\omega_0^2 - \omega^2)A}{F_0}; \quad \sin\phi = \frac{-m\omega\gamma A}{F_0} < 0 \tag{4.9}$$

$$\tan\phi = -\frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{4.10}$$

Knowing $\tan\phi$ is not enough to tell us what ϕ is because, for example, $\tan\frac{\pi}{3} = \tan -\frac{2\pi}{3}$. However, because m, ω, γ, A, and F_0 are all positive, we know that $\sin\phi$ is negative $\implies -\pi \leq \phi \leq 0$, which is enough extra information to find ϕ given $\tan\phi$. $\phi < 0$ means that the displacement lags *behind* the driving force, as one might expect (stimulus precedes response. There is a delay of ϕ/ω seconds between the driving force changing direction and the displacement changing direction.

*To persuade yourself that $\tilde{x}(t)$ can't have a frequency different from ω, try substituting $\tilde{x}(t) = Ae^{i(\omega' t+\phi)}$ and dividing across by $e^{-i\omega' t}$. You'll find that one side of the Equation is time dependent and the other is not.

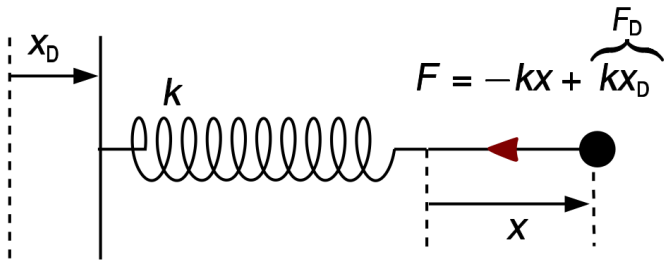


Figure 4.2: One way to apply a driving force to the mass-and-spring oscillator is to move the other end of the spring as $x_D(t) = \dfrac{F_0 \cos \omega t}{k}$

It is now straightforward to show (e.g. using the identity $\sin^2 \phi + cos^2\phi = 1$) that

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \tag{4.11}$$

Therefore, a solution of Equation 4.3 is the real part of $\tilde{x}(t)$, which is

$$x_s(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \cos(\omega t + \phi) \tag{4.12}$$

However, x_s is not the general solution, but a particular solution (often referred to in mathematics as the **particular integral**). The general solution is $x_s + x_t$ where x_t is the general solution of the (homogeneous) damped oscillator equation. In mathematics x_t is often referred to as the **complementary function**. We refer to it as the **transient** and discuss it in Section 4.3.

We refer to x_s as the **steady state** solution. It is the state of the oscillator a very long time after the driving force has been switched on, when the sum of the driving and the damping forces has settled to a constant times the displacement.

$$F_d + F_D = -m\gamma\dot{x} + F_0 \cos \omega t = -m(\omega^2 - \omega_0^2)x$$

Resonance

The steady state amplitude is plotted as a function of ω in Figure 4.3. If we examine Equation 4.11, we can observe that, if there isn't any damping ($\gamma = 0$), the amplitude diverges when $\omega \to \omega_0$. This is because, if the driving force is at the natural frequency of the spring, it is always in phase with the motion of the mass. Therefore, the driving force continuously reinforces and amplifies the oscillation.

Of course, this may not be true at the beginning. It is perfectly possible for the displacement of the oscillator from equilibrium to be parallel, rather than antiparallel, to the driving force at the instant when the driving force is switched on. In this case, the oscillator would be out of phase with the driving force. However, after a time, it would be forced into phase and its

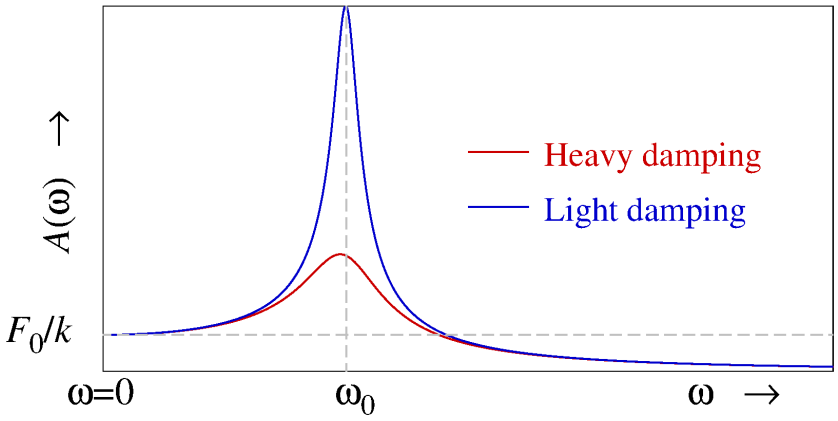


Figure 4.3: Steady state amplitude as a function of driving frequency ω for a forced oscillator of natural angular frequency ω_0.

amplitude would then continue to grow unchecked to an arbitrarily high value. This strong coupling between the driving force and the oscillator is known as **resonance**. More generally, resonance refers to the enhanced coupling (transfer of energy) between two bodies that share a common frequency. The enhancement is a consequence of their synchronous motion which allows them to optimally drive or impede one another.

Throughout nature it is the case that two bodies whose dynamics have different characteristic timescales tend to couple less efficiently due to cancellation of effects: sometimes they impede one another and sometimes they reinforce one another and so the time average of the coupling vanishes.

Damping

If, as is always the case in a real system, the motion is damped ($\gamma \neq 0$), the amplitude can't continue to grow unchecked because, as the amplitude of an oscillation grows, so does the average speed and hence the average magnitude of the damping force $\gamma\dot{x}$. This prevents the resonance between the oscillator and the driving force from causing the amplitude to diverge.

The dissipation also causes a lag between the driving force and the oscillator's response to it, which manifests mathematically as the phase difference ϕ between $x(t)$ and $F_D(t) = F_0 \cos \omega t$. This means that for a portion of each oscillation period the velocity is parallel to the driving force and for another portion of the period they are antiparallel.

Effect of anharmonicity on resonance

Let's consider, once again, the example of the orange rolling in the bowl. Recall what happens when the oscillation amplitude gets large: the harmonic approximation breaks down because higher order terms in the Taylor expansion of the potential energy become relevant. As a consequence, the frequency changes, as illustrated in Figure 1.11. Now let us suppose that the orange is driven by moving the table on which the bowl rests back and forth sinusoidally. If it is driven at the frequency of small oscillations about the bottom of the bowl, resonance causes its amplitude to increase very quickly, whereupon its frequency

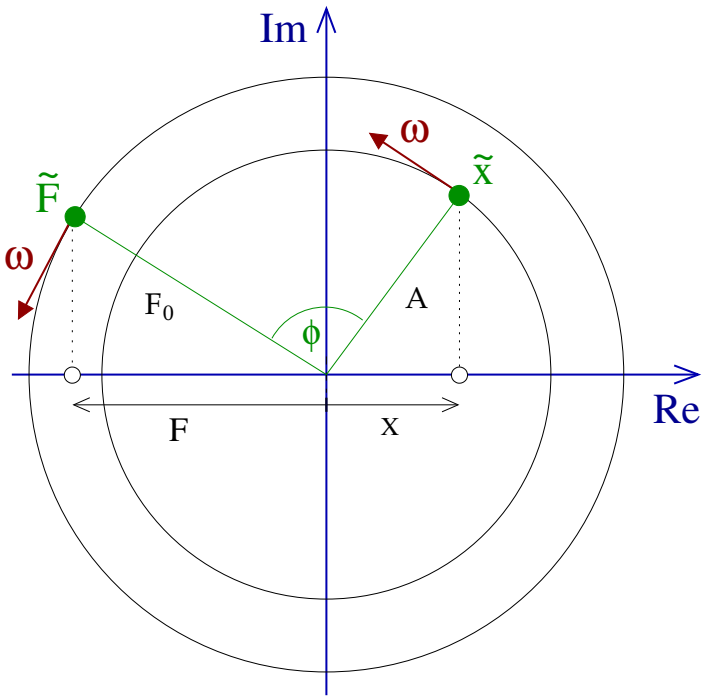


Figure 4.4: $\tilde{F}(t) = F_0 e^{i\omega t}$ and $\tilde{x}(t) = Ae^{i(\omega t - \phi)}$ represented in the complex plane as "phasors". Both move in a circle with angular velocity ω, but $\tilde{x}$ lags behind $\tilde{F}$ by an angle ϕ and a time ϕ/ω. This lag means that, for a portion of each period, the driving force opposes the motion.

changes and it falls out of resonance and out of phase with the driving force. In this way, anharmonicity impedes resonant excitation.

4.3 Transients

How the system reaches its steady state can vary greatly depending on its initial conditions. Any solution of the homogeneous equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{4.13}$$

can be added to the solution of

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \tag{4.14}$$

to get the general solution. The homogeneous equation is the damped oscillator equation of motion and its general solution is

$$x_t(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \theta) \tag{4.15}$$

where A_0 and θ are determined by initial conditions. We are assuming that the damping is not strong, i.e. $\dfrac{\gamma^2}{4} < \omega_0^2$.

The complete solution is

$$x(t) = x_s(t) + x_t(t) = \underbrace{A\cos(\omega t + \phi)}_{\text{steady state}} + \underbrace{A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \theta)}_{\text{transient}} \tag{4.16}$$

where

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \tag{4.11}$$

$$\tan\phi = -\frac{\omega\gamma}{\omega_0^2 - \omega^2} \tag{4.10}$$

$$\omega_d^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{3.12}$$

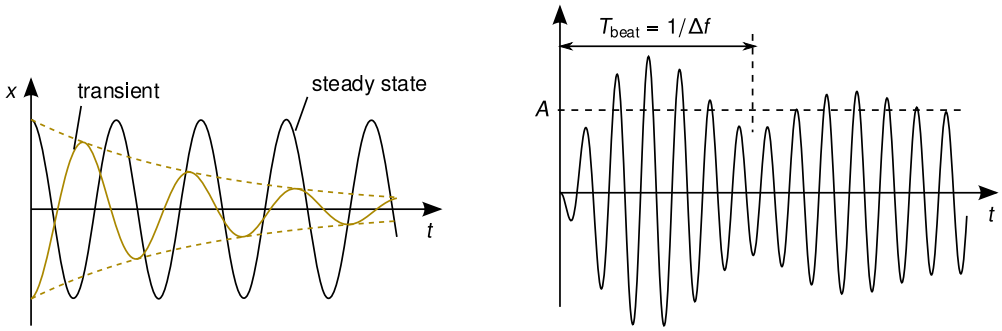


Figure 4.5: Left: The steady state (x_s) and the transient (x_t) parts of the solution of the equation of motion of the forced damped oscillator. Right: The net displacement $x(t) = x_s(t) + x_t(t)$ of as a function of time for a near-resonant driving frequency and a particular choice of initial conditions. Unless the oscillator has a $Q-$factor close to 0.5, $\omega_d \approx \omega_0 \Rightarrow \omega_d \approx \omega$ and the superposition of the steady state and transient solutions exhibits beating until the transient dies.

4.4 The steady state in different frequency regimes

We can identify three distinct frequency regimes, representing different kinds of steady state behaviour of the forced damped oscillator. Let's consider how we expect the system to behave in steady state - after the transient has passed, having been extinguished by the exponential decay factor $e^{-\gamma t/2}$ in Equation 4.16.

If there wasn't a driving force, the damping force would eventually cause the amplitude to decay to zero. Therefore, it should not be surprising that, at all frequencies, the amplitude of *steady state* oscillations is proportional to the magnitude of the driving force, F_0. However, the sensitivities to the driving frequency (ω), the mass (m), the spring (k), and the damping strength (γ) are different in different frequency regimes.

4.4.1 Stiffness controlled vibrations ($\omega \ll \omega_0$)

In the limit $\omega/\omega_0 \to 0$, the denominators of Equations 4.11 and 4.10 become ω_0^2. The steady state amplitude and phase are

$$A \approx \frac{F_0/m}{\omega_0^2} = \frac{F_0}{k}$$
$$\tan\phi \approx -\frac{\omega\gamma}{\omega_0^2} \approx 0 \Rightarrow \phi \approx 0 \quad (\phi < 0)$$

In this limit, the amplitude is independent of the damping constant (γ), the mass (m), and the driving frequency(ω). Although it depends on k, the amplitude is also independent of the natural frequency ω_0. This can be understood as follows: the driving force changes so slowly that, on its typical response timescale ($\sim \omega_0^{-1} \ll \omega^{-1}$), the mass simply sees it as a constant force and responds to it by moving to the minimum of potential energy. On the timescale of the driving force, the mass responds instantaneously to changes in the force. Therefore, there is no lag between F_D and x ($\Rightarrow \phi = 0$). The position at each time is simply given by Hooke's Law $x(t) = F_D(t)/k$ and so the maximum displacement is $A = F_0/k$.

4.4.2 Mass controlled vibrations ($\omega \gg \omega_0$)

On the other hand, if the driving force has a very high frequency it reverses its direction before the mass has had a chance to respond to it. The acceleration of the mass by the driving force, $F_D(t)/m$ is too small in magnitude for it to have any significant impact on the motion before it reverses its direction. Therefore, on the timescale of response ($\sim \omega_0^{-1}$) of the mass its effect averages to zero. The steady state amplitude and phase are

$$A \approx \frac{F_0/m}{\omega^2} \tag{4.17}$$
$$\tan\phi = \frac{\gamma}{\omega}\left(\frac{1}{1-\frac{\omega_0^2}{\omega^2}}\right) \approx \frac{\gamma}{\omega}\left(1+\frac{\omega_0^2}{\omega^2}\right) \approx 0$$
$$\Rightarrow \phi \approx -\pi \tag{4.18}$$

4.4.3 Resistance limited vibrations ($\omega = \omega_0$)

When the driving frequency is at the natural frequency, the only thing limiting the growth of the amplitude is the damping, (γ). This is a dissipative resistance to motion. In the context of electrical circuits, which we'll discuss in Section 4.5, the dissipative component is called a resistor (R) or a resistance.

$$A = \frac{F_0/m}{\omega_0\gamma} = \frac{F_0}{k}Q \tag{4.19}$$
$$\phi \approx -\frac{\pi}{2} \tag{4.20}$$

Notice that ϕ is never positive and so $x(t)$ always *lags behind* $F_D(t)$. The stimulus precedes the response.

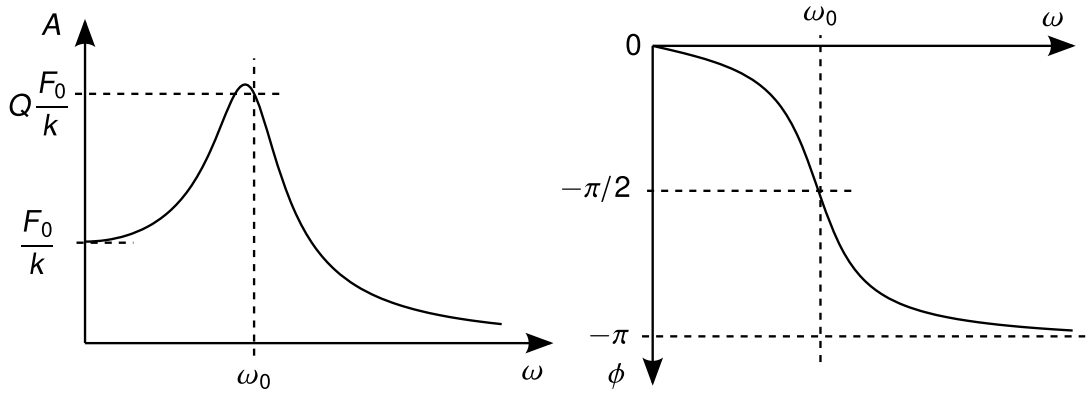


Figure 4.6: Steady state amplitude (A) and phase lag (ϕ) as a function of driving frequency (ω) for a driven damped oscillator.

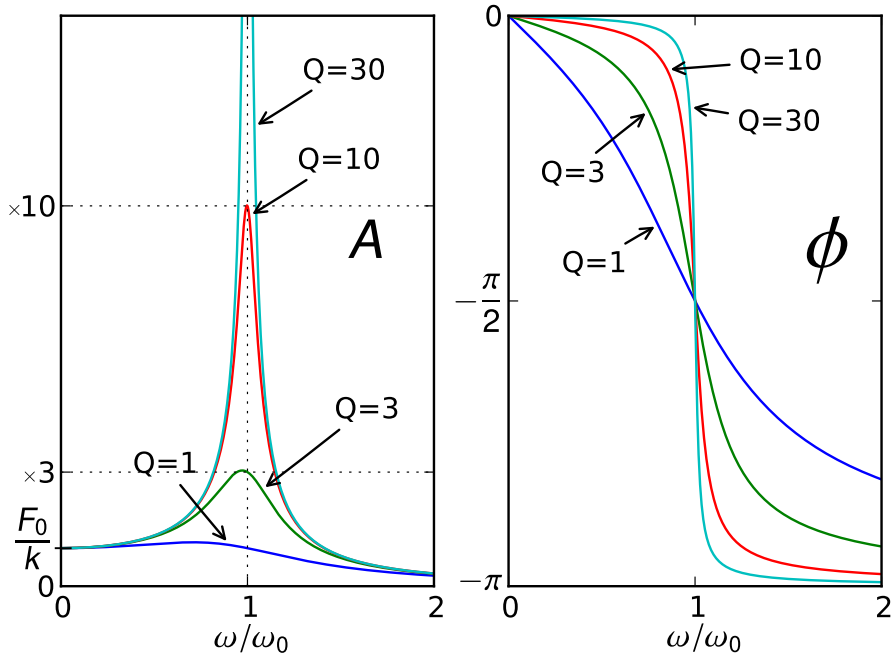


Figure 4.7: Steady state amplitude (A) and phase lag (ϕ) as a function of driving frequency (ω) for driven damped oscillators with different $Q-$factors.

Exercise:

Starting from Equation 4.11, show that the driving frequency for which the steady-state amplitude of the forced damped oscillator is a maximum, is

$$\omega_{max} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \tag{4.21}$$

and show that the maximum amplitude is

$$A(\omega_{max}) = A(\omega_0)\frac{1}{\sqrt{1 - 1/(4Q^2)}} \tag{4.22}$$

4.5 Example: Electrical resonance circuits

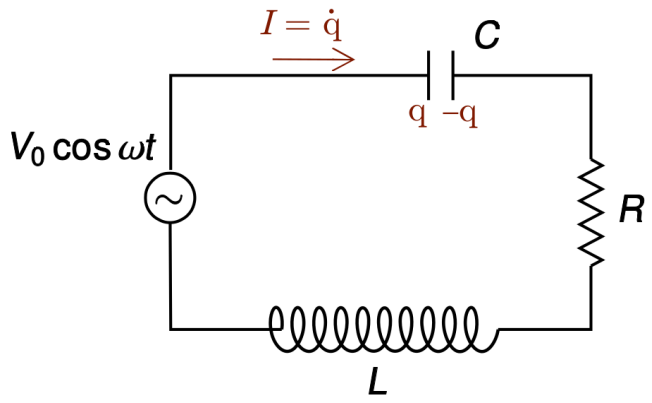


Figure 4.8: An electrical resonance circuit, also known as a driven LCR circuit, consists of a capacitor, resistor, and inductor connected in series with a sinusoidal AC power source.

Consider the circuit depicted in Figure 4.8. Let us choose the clockwise direction to be positive, meaning that the current is considered positive if it flows clockwise and negative if it flows anti-clockwise. The potential change across each component (L, C, or R) is considered positive[negative] if the potential increases[decreases] when traversing it in the clockwise direction.

Kirchoff's voltage law

The sum of the potential changes around any closed loop in a circuit is zero. In the diagram of the circuit above, which consists of only one loop, we get

$$-V_C - V_R - V_L + V(t) = 0 \tag{4.23}$$

where V_C, V_R, and V_L, are the potential *drops* across the capacitor, the resistor, and the inductor, respectively, and $V(t) = V_0 \cos \omega t$ is the electromotive force (EMF) from the alternating power source, which is positive when it is driving the current clockwise and negative when it is driving the current anticlockwise.

Capacitor

A capacitor stores positive and negative electric charges, separated from one another by a non-conducting (dielectric) region. Referring to the circuit diagram (Figure 4.8), let's denote the charge on the left hand side of the capacitor by q, meaning that the charge on the right hand side is $-q$. When current is flowing clockwise, the charge on the left hand side is increasing. The capacitance C is the magnitude of the ratio of the stored charge to the potential drop across the capacitor, V_C. With our sign convention

$$C = \left| \frac{q}{V_C} \right| \Rightarrow V_C = \frac{q}{C} \tag{4.24}$$

because, when $q > 0$, there is a potential drop across the capacitor in the clockwise direction and, when $q < 0$, there is a potential increase across the capacitor, which is equivalent to a negative potential drop.

Resistor

The potential drop across an imperfect conductor ("resistor") is directly proportional to the current flowing through it (Ohm's Law)

$$I = \frac{V_R}{R} \Rightarrow V_R = RI \tag{4.25}$$

where the constant of proportionality R is known as the **resistance**.

Inductor

An inductor is a conducting coil. The magnetic flux ϕ_B through each loop of the coil is proportional to the current flowing in it.

$$\phi_B \propto I$$

Faraday's law of induction tells us that a changing flux in the loop produces an EMF opposing the current of $-\frac{d\phi_B}{dt} \propto -\frac{dI}{dt}$. V_L is the potential drop across the inductor, which has contributions from all loops of the coil and is the negative of the induced EMF. It is proportional to the rate of change of the current flowing through it and the proportionality constant, L, is known as the **inductance**.

$$V_L = L\frac{dI}{dt} \tag{4.26}$$

Equation of motion

The state of the circuit at any instant can be specified by the charge on the capacitor, $q(t)$, and the current $I(t)$, which equals the rate of change of charge on the capacitor $\dot{q}(t)$.

$$I = \frac{dq}{dt} \tag{4.27}$$

Substituting Equations 4.24, 4.25, 4.26, and 4.27 into Equation 4.23 gives

$$\begin{aligned} -\frac{q}{C} - RI - L\frac{dI}{dt} + V_0 \cos\omega t = 0 \\ \Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \cos\omega t \end{aligned}$$

Let's rearrange this and compare it to the forced oscillator equation (Equation 4.3)

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \tag{4.3}$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_0}{L}\cos\omega t \tag{4.28}$$

Mechanical oscillator				Electrical oscillator
displacement	x	$\Leftrightarrow$	q	charge
velocity	$v = \dot{x}$	$\Leftrightarrow$	$I = \dot{q}$	current
mass	m	$\Leftrightarrow$	L	inductance
damping constant	b	$\Leftrightarrow$	R	resistance
spring constant	k	$\Leftrightarrow$	$1/C$	$1/$capacitance
force amplitude	F_0	$\Leftrightarrow$	V_0	EMF amplitude

Table 4.1: Physical characteristics and state variables of a forced mechanical oscillator and the corresponding quantities for the electrical LCR oscillator depicted in Figure 4.8

It is clear that, mathematically, there is no difference between the equations describing the forced damped mechanical oscillator and the electrical resonance circuit comprising an AC power source, a capacitor, a resistor, and an inductor in series. This is more commonly known as a driven **LCR circuit**.

We can write the driven LCR equation as

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = \frac{V_0}{L}\cos\omega t \tag{4.29}$$

where $\gamma = \dfrac{R}{L}$ and $\omega_0 = \dfrac{1}{\sqrt{LC}}$. For each quantity in the mechanical oscillator there is a corresponding quantity in the electrical oscillator which plays an equivalent role. The correspondences are listed in Table 4.1.

Note that, if we only had an inductor and a capacitor (i.e. $R = 0$, $V_0 = 0$) we'd have the electrical analogue of the simple harmonic oscillator - an LC circuit. The equation of motion would be

$$\begin{aligned} \ddot{q} + \frac{1}{LC}q &= 0 \\ \Rightarrow \ddot{q} + \omega_0^2 q &= 0 \end{aligned} \tag{4.30}$$

If we had an inductor, a capacitor, and a resistor ($R \neq 0$, $V_0 = 0$) we'd have the electrical analogue of the damped harmonic oscillator - an (undriven) LCR circuit.

$$\begin{aligned} \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q &= 0 \\ \Rightarrow \ddot{q} + \gamma\dot{q} + \omega_0^2 q &= 0 \end{aligned} \tag{4.31}$$

The solution of the driven LCR equation of motion is mathematically identical to that for the mechanical oscillator. It has a steady state and a transient part. You may have noticed the transient when you switch on a TV (particularly an old one): the sound and picture don't appear immediately. There is a slight delay before steady state is reached in the TV's circuits.

4.6 Power dissipation and absorption

The damping force describes dissipation of the oscillator's energy. Forcing/driving involves the absorption of energy by the oscillator. When steady state is reached, the average energy of the oscillator has settled to a constant value so that the energy absorbed equals the energy dissipated. The transient is a period at the beginning when the amplitude and phase are adjusting themselves so that dissipation and absorption of energy are balanced.

The work done by the oscillation against the damping force as the displacement changes by Δx is

$$\Delta W = -F_\mathrm{d}\Delta x$$

The rate at which work is done against the damping force is

$$\text{Power,} \quad P = \frac{dW}{dt} = -F_\mathrm{d}\frac{dx}{dt} = -F_\mathrm{d}v$$

The velocity is $v = \dot{x} = -\omega A\sin(\omega t + \phi)$ and the damping force is given by $F_d = -bv$. Substituting for these gives

$$P = bv^2 = b\omega^2 A^2\sin^2(\omega t + \phi)$$

Taking the time average (over one cycle is sufficient) gives

$$\begin{aligned} P_\mathrm{av} &= \frac{1}{T}\int_t^{t+T} P dt \\ &= \frac{1}{2}b\omega^2 A^2 \end{aligned}$$

Subsituting for A (Equation 4.11) and using $b/m = \gamma$ gives

$$P_\mathrm{av} = \frac{1}{2}\frac{F_0^2\gamma}{m}\frac{\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \tag{4.32}$$

If the system is in a steady state, this is equivalent to the work done by the driving force over one complete cycle.

$$\omega \ll \omega_0: \qquad P_\mathrm{av} \approx \frac{1}{2}\frac{F_0^2\gamma}{m}\frac{\omega^2}{\omega_0^4} \sim \omega^2 \qquad \text{(stiffness controlled)}$$

$$\omega \gg \omega_0: \qquad P_\mathrm{av} \approx \frac{1}{2}\frac{F_0^2\gamma}{m}\frac{1}{\omega^2} \sim \frac{1}{\omega^2} \qquad \text{(mass controlled)}$$

4.6.1 Maximum power

To find the maximum, over all driving frequencies, of the power, rewrite P_av as

$$P_\mathrm{av} = \frac{1}{2}\frac{F_0^2\gamma}{m}\frac{1}{\left(\omega_0{}^2/\omega-\omega\right)^2+\gamma^2}$$

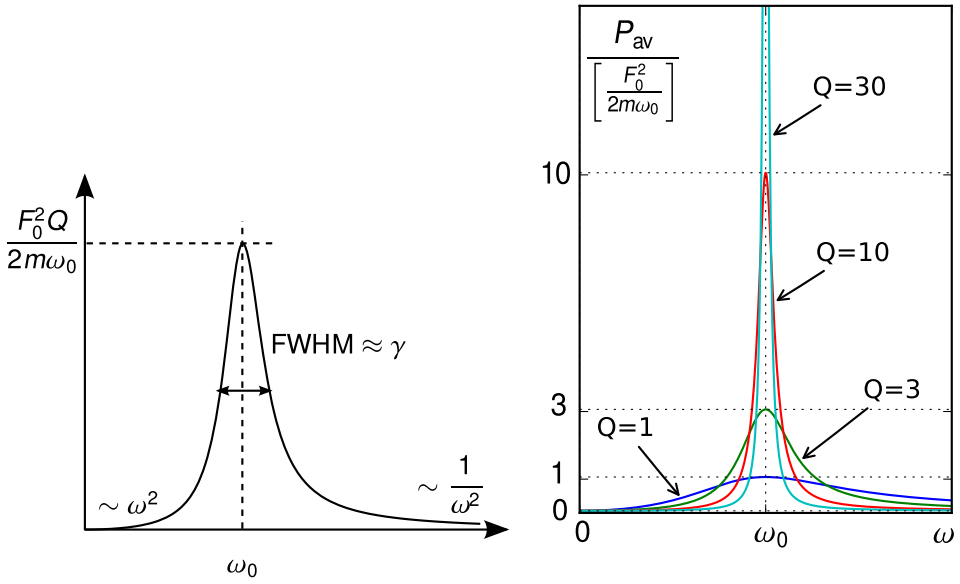


Figure 4.9: Left: Average over one period of the power dissipated by the oscillator (P_{av}) as a function of ω. Right: $P_{av}(\omega)$ for several values of Q relative to its maximum value for $Q = 1$.

from which we can see that the denominator is minimized for $\left({\omega_0}^2/\omega - \omega\right) = 0$, giving the maximum power dissipation at $\omega = \omega_0$,

$$P_{\text{av}}\Big|_{\omega=\omega_0} = \frac{1}{2}\frac{F_0^2\gamma}{m}\frac{1}{\gamma^2} = \frac{1}{2}\frac{F_0^2}{\gamma m} = \frac{1}{2}\frac{F_0^2}{b}$$
$$= \frac{F_0^2 Q}{2m\omega_0}$$

P_{av} falls to half its maximum value for $(\omega^2 - \omega_0^2) = \pm\omega\gamma$. For reasonable Q ($\gamma \ll \omega_0$),

$$(\omega^2 - \omega_0^2) = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega(\omega - \omega_0)$$
$$\Rightarrow \quad \omega \approx \omega_0 \pm \frac{\gamma}{2} \qquad \gamma \ll \omega_0$$

The full-width-at-half-maximum FWHM $\approx \gamma$. We therefore call γ the **width**. In terms of the Q-factor, FWHM $\approx \omega_0/Q$.

The function $P_{\text{av}}(\omega)$ can be interpreted as a power absorption curve for the system when driven by an oscillating force of angular frequency ω. The strength of the absorption peak (P_{av} at $\omega = \omega_0$) is proportional to Q.