EM Section 3: Gauss's Law

In principle it is always possible to find the electric field from an arrangement of static charges by use of Coulomb's law, but the more awkward the geometry of the situation the less trivial the mathematics – often involving integral calculus – becomes. Gauss's law for is a law that follows as a result of Coulomb's law that provides a method of finding the electric field for certain situations in a much neater way. It can also be used to deduce an arrangement of charges from the knowledge of an electric field. It is also one of Maxwell's equations.

This chapter deduces the law using Coulomb's law and a definition of a quantity known as electric flux then goes on to provide some classic examples before seeing how the law applies to gravity.

3.1 Electric flux

Uniform electric field definition

Consider a uniform electric field $\boldsymbol{E}$ passing through an area $\boldsymbol{A}$ at an angle θ as represented by its field lines in figure 3.1:

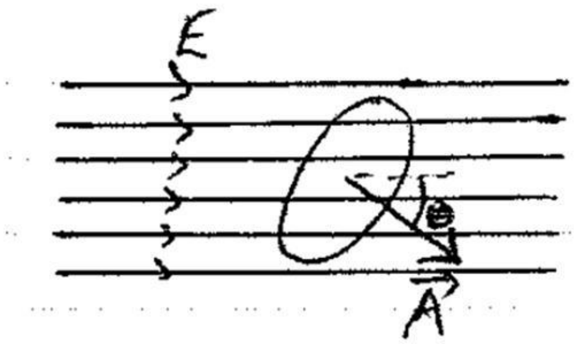

Figure 3.1 A uniform electric field intersecting an area at an angle to it

The electric flux, Φ (Greek letter phi) through the area is defined as the dot product of the electric field with the area, i.e.

$$\Phi = \boldsymbol{E} \cdot \boldsymbol{A} = EA\cos\theta \qquad \textbf{Equation 3.1}$$

Electric flux is a scalar quantity with SI units of $\mathrm{Nm^2C^{-1}}$ or Vm. The base SI units are $\mathrm{kgm^3s^{-3}A^{-1}}$. It is acceptable to use any of the standard SI prefixes for large and small electric fluxes though standard form is fine as well.

Note that the area in question can be imaginary – and in fact often is when invoking the use of electric fluxes in physics. It certainly does not have to be a physical, real sheet of any kind of material.

So if an area is aligned perpendicular to a uniform field the flux through the area takes a maximum value of EA and if either the area or field is rotated so the angle between the vectors increases the flux through the area diminishes according to equation 3.1 so when the vectors are perpendicular (i.e. the sheet is aligned parallel to the field lines) the field becomes exactly zero. And if the rotation continues the flux becomes increasingly negative until the angle between the two is $180°$ - the vectors are antiparallel and the flux takes a minimum value of $-EA$.

So if a flat sheet of dimensions $2.0\text{ cm} \times 3.0\text{ cm}$ is aligned a $40°$ to a uniform electric field of strength 45 Vm^{-1} the angle between the two vectors is $50°$ and the electric flux through the area is $EA\cos\theta = 45 \times 0.02 \times 0.03 \times \cos 50° \approx 17\text{ mVm}$.

A generalised definition for non-uniform electric fields and areas

If a non-uniform electric field passes through a non-uniform, non-planar (i.e. curved) area then the formula for the electric flux through the area must be generalised. Using equation 3.1, the infinitesimal flux, $\delta\Phi$, through an infinitesimal vector area, $\delta\boldsymbol{A}$, is given by $\delta\Phi = \boldsymbol{E}\cdot\delta\boldsymbol{A}$ where $\boldsymbol{E}$ is the field strength at the point in question. To find the flux through the whole area, the sum of each individual contribution of $\boldsymbol{E}\cdot\delta\boldsymbol{A}$ must be calculated. In the limit that $\delta A \to 0$ this becomes the surface integral

$$\Phi = \iint_S \boldsymbol{E}\cdot d\boldsymbol{A} \qquad \textbf{Equation 3.2}$$

hence the flux of any area for any field can be computed. For an arbitrary field with an arbitrary area the integral in unlikely to have an exact analytical solution and numerical approximations will be required. But for symmetrical charge distributions and mathematically friendly areas then exact solutions may be possible.

Flux due to a point charge through a sphere with the charge at its centre

So for a sphere surrounding a positive point charge q with the charge at its centre: if the sphere has radius R the magnitude of the field strength at any point on the sphere is given by $\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$ with the direction always parallel to the element of area it crosses. Equation 3.2 therefore simplifies to

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} \oiint_{Sphere\ of\ radius\ R} \hat{\boldsymbol{n}}\cdot d\boldsymbol{A}$$

As the surface integral over the whole sphere is simply the surface area of the sphere i.e.

$$\oiint_{Sphere\ of\ radius\ R} \hat{\boldsymbol{n}}\cdot d\boldsymbol{A} = 4\pi R^2$$

the flux through the sphere is simply $\frac{Q}{\epsilon_0}$.

And the generalisation of this

It turns out that this simple rule is quite general:

For any net charge Q the flux through any closed surface covering the charge is given by $\frac{Q}{\epsilon_0}$.

This wonderful rule – Gauss's law – is a consequence of the inverse square law and is proved in the next section.

3.2 Gauss's law in electrostatics

Gauss's law states that the net electric flux through any closed surface is directly proportional to the charge contained within the surface. In the SI system of units the flux is equal to the charge contained divided by ϵ_0. In equation form this can be written:

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\epsilon_0} \qquad \textbf{Equation 3.3}$$

The law is useful for two directions of inquiry:

i) If an electric field is known around a surface then the charge enclosed can be determined

ii) If a charge inside a surface is known the then the electric field can be found

We will study examples of (ii) in some detail and the discussion of (i) will be briefer. First however, a proof of the law for a general case is provided.

Proof

There are various ways of proving Gauss' law, depending on which form is being sought (see section 3.3 for another form) and which route is being taken to prove the law. In essence the law turns out to be equivalent to Coulomb's law. One can start with Coulomb's law to get to equation 3.3 or equally well start with equation 3.3 and show that Coulomb's law follows from it. In these notes we just show the one proof of equation 13.3 from Coulomb's law:

Consider a point charge Q surrounded by an arbitrary surface S as shown in figure 3.2

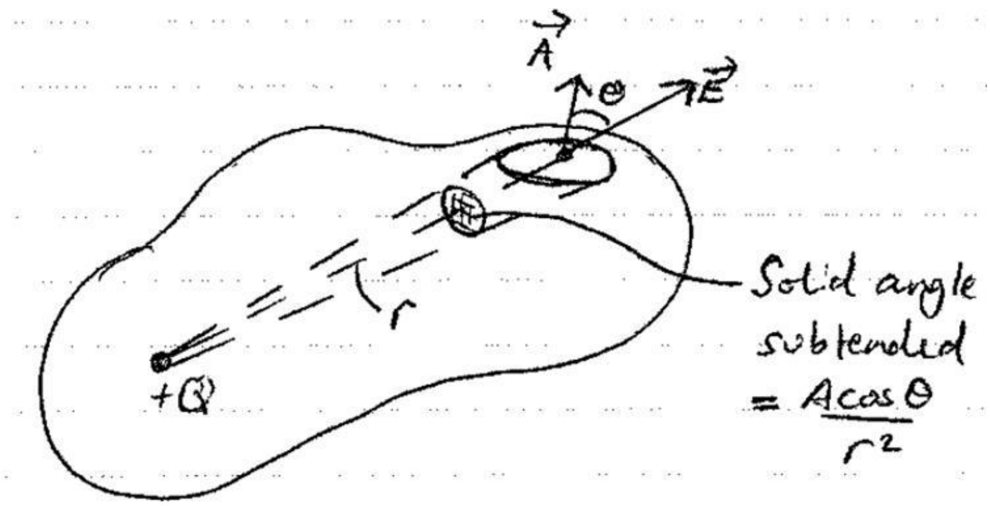

Figure 3.2 A point charge enclosed by a general surface

Consider the flux due to the point charge through an element of area $\delta\boldsymbol{A}$ as indicated on the diagram:

The flux of the electric field due to the point charge through this area is given in general by $\boldsymbol{E}.\delta\boldsymbol{A}$.

This flux can be written as

$$E.\delta A.\cos\theta$$

And if the charge is a linear distance r from the element of area, Coulomb's law gives the flux as

$$\frac{Q\cos\theta\,\delta A}{4\pi\epsilon_0 r^2}$$

Now consider the element of solid angle subtended by the element of area. It has a value given by

$$\delta\Omega = \frac{\delta A\cos\theta}{r^2}$$

Thus the flux through the area element is

$$\frac{Q\delta\Omega}{4\pi\epsilon_0}$$

So to compute the flux through the entire area this can be written as the integral

$$\oiint_S \boldsymbol{E}\cdot d\boldsymbol{A} = \oiint_S \frac{Q}{4\pi\epsilon_0} d\Omega = \frac{Q}{4\pi\epsilon_0}\oiint_S d\Omega$$

And as the integral of a solid angle over a whole surface is exactly 4π this gives

$$\oiint_S \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{Q}{\epsilon_0}$$

This is the flux for one point charge. It therefore follows from the principle of superposition that the total flux is given by the sum of all the point charges enclose i.e.

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

This proof is completely general; it applies to any charge distribution enclosed by any closed surface.

3.3 Different forms of the law

Gauss's law as stated by equation 3.3 is sometimes referred to as the integral form of Gauss's law. Using the laws of vector calculus – in particular the divergence theorem – this can be written in a differential form as follows:

The divergence theorem for the left hand side of equation 3.3 states that

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \iiint_V (\nabla \cdot \boldsymbol{E}) dV$$

And as the charge enclosed by a volume can be written as the volume integral of the charge density, ρ, as

$$Q = \iiint_V \rho dV$$

equation 3.3 can be rewritten as

$$\iiint_V (\nabla \cdot \boldsymbol{E}) dV = \iiint_V \frac{\rho}{\epsilon_0} dV$$

The condition for this to be true is that the integrands be equal. Hence it can be stated that:

$$\nabla . \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$ **Equation 3.4**

i.e. the divergence of the electric field equals the total charge density enclosed.

Note that Coulomb's law, equation 3.3 and equation 3.4 are all different ways of stating the inverse square law. Any one can be proved from any other.

Or almost can: to prove Coulomb's law from Gauss's law the assumption must be made that the field from a point charge is spherically symmetric, which cannot be elucidated from the mathematical statements of Gauss's law in the absence of any other information.

3.4 Examples on finding the field from a charge distribution

The electric field due to any charge distribution, can, in principle, be deduced from equation 3.3. The simplest charge "distribution" to do this with is simply a point charge, as has been seen earlier in the chapter. Other spherically symmetric charge distributions are addressed in the end of chapter problems and the related tutorial discussion problem.

For other charge distributions finding an exact analytical solution can be difficult, or even impossible, but provided a situation is highly symmetrical a solution may well be possible. There are two particular well-known and important charge distribution/field combinations that all physicists should know and they are given here:

The field due to an infinite uniform line of charge

Consider a straight, infinitely long, line of charge with uniform charge per unit length $\lambda\ \mathrm{Cm}^{-1}$ aligned along the x- axis. What is the field as a function of distance from the line?

Consider a point at a distance r from the line. As the line of charge is completely symmetric and infinite in length, there can be no component of the field parallel to line at the point; the field must act perpendicular to line of charge at all points, pointing towards the line for a negatively charged line and away for a positively charged line.

The trick to using Gauss's law to compute the field strength is consider an imaginary cylinder of radius r and length h sharing its axis with the line of charge as shown in figure 3.3:

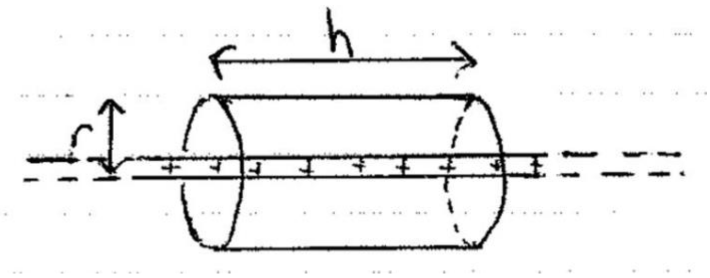

Figure 3.3: An infinite line of charge with a Gaussian cylinder enclosing a section

This kind of imaginary volume enclosing a charge distribution is known as a Gaussian surface.

The cylinder encloses a charge of total value λh C and so Gauss's law says that the flux through the cylinder is given by

$$\oiint_{Whole\ cyclinder} \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{\lambda h}{\epsilon_0}$$

But how to calculate the left hand side of the equation? First consider the ends of the cylinder. The flux through the ends is just zero, and there cannot be any component of the field in that direction. The flux through the remainder is also easy to compute once it is recognised that it must be uniform over the whole area, and perpendicular at all points. If the magnitude of the field at the surface is E, and the area it goes through is $2\pi rh$ then the total flux is given by $2\pi rhE$.

Hence Gauss's law for the cylinder gives

$$2\pi rhE = \frac{\lambda h}{\epsilon_0}$$

which gives an expression for the magnitude of the field due to an infinite line of charge of

$$E_{line} = \frac{\lambda}{2\pi\epsilon_0 r}$$

This may be a familiar result, showing that the electrostatic force due to a line of charge is inversely proportional to the distance i.e. it is less severe a drop-off that the inverse square relation of a point charge.

Notice how much simpler this was than finding the answer in problem 2.2!

The field due to an infinite sheet of charge

Now consider an infinite plane of charge of charge density σ Cm^{-2}. What is the field in this case?

Again, consider a point a distance r above the plane and recognise that there can be no component of the field parallel to the plane for the symmetric case therefore the field acts directly towards or away from the nearest point on the plane depending on the sign of the charge density.

Once again, a Gaussian surface is selected to render the problem simple by the use of Gauss's law, and once again a cylinder is selected. The orientation required is shown in figure 3.4:

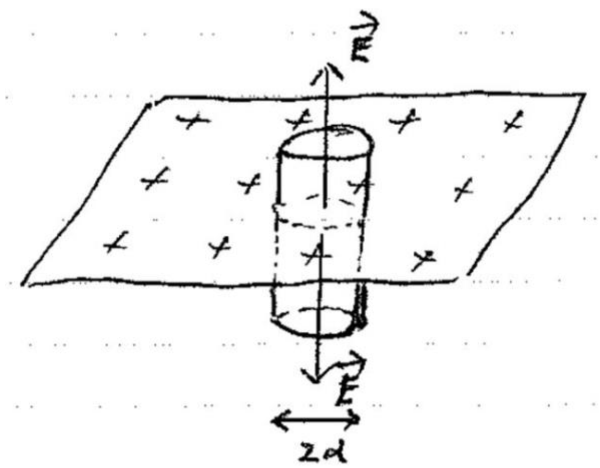

Figure 3.4 A Gaussian pillbox around a sheet of charge

The cylinder in this setup is often referred to as a "Gaussian pillbox".

The analysis proceeds in a similar way as with the line of charge with one key difference: in this case the ends of the cylinder have non-zero flux and the flux through the curved surface is zero.

If the cylinder has radius d then the charge enclosed by the pillbox is $\pi d^2\sigma$ and the flux through the ends is $E.\pi d^2$ for each end for a total flux of $E2.\pi d^2$.

So using Gauss's law,

$$\oiint_{Whole\ cylinder} \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\epsilon_0}$$

becomes

$$E2.\pi d^2 = \frac{\pi d^2\sigma}{\epsilon_0}$$

which gives an expression for the field due to the sheet as

$$E_{sheet} = \frac{\sigma}{2\epsilon_0}$$

Again, this is possibly a familiar result showing that the field strength is independent of position.

3.5 More on fields inside a conductor

In section 2.5 the field inside a conductor at equilibrium is always zero with the electrostatic field at all points on the surface always acting perpendicular to the plane of the surface at a point. Gauss's law can be used to further conclude that any net charge on a conductor resides at its surface at equilibrium.

Consider a conductor of any shape with a net charged placed inside. In a perfect conductor the charges moves so that the field inside the conductor is zero. This happens with a matter of picoseconds. Now consider forming a Gaussian surface of any kind with the surface entirely contained within the conductor. If the Gaussian surface contained any charge then there would be a flux through it, which would create a field. As the field is zero at all points with the surface the only way for there to never be a flux is for all the charge inside a conductor to reside at its surface.

3.6 Gauss's law with gravitational fields

Though surprisingly less well known than Gauss's law for electrostatics, as classical gravity also follows an inverse square law, there are perfectly analogous equations for Gauss's law for gravity i.e:

Integral form: $\oiint_S \ \boldsymbol{g} \cdot d\boldsymbol{A} = -4\pi GM$

Differential form $\nabla . \boldsymbol{g} = -4\pi G\rho$

Where $M$is an enclosed mass and ρ is the density. Notice how with Gauss's law for electrostatics the clumsy 4π factor from the force equations vanishes but manifests itself in Gauss's law for gravity.