EM Section 5: Capacitance

Most of you will have heard of capacitors. Some of you will be very familiar with them and have practical knowledge of their use in circuits. They are also of great importance in developing a theoretical knowledge of electromagnetism and provide an important step en route to developing Maxwell's equations in their full form.

5.1 Capacitance and capacitors

Consider two identical parallel conducting plates with an area of magnitude A held a distance d apart with a vacuum in between. The plates are given equal and opposite charges. A schematic diagram of the set up is shown in figure 5.1

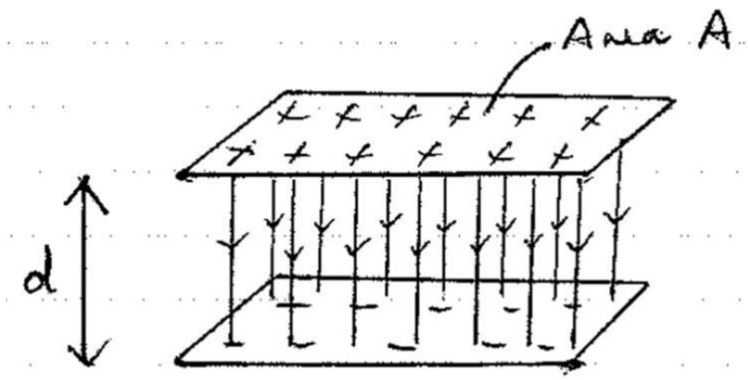

Figure 5.1 Two charged plates held a fixed distance apart

The charges spread themselves evenly over the plates so they have charge densities of $\pm\sigma$ and a uniform electric field exists between the plates. From section 3 the use of Gauss's law means the field has magnitude $\frac{\sigma}{\epsilon_0}$, and if the potential difference between the plates is V this field strength can also be equated to $\frac{V}{d}$ according to section 4.1.

Equating these two expressions gives $\frac{\sigma}{\epsilon_0} = \frac{V}{d}$ and if the charge density is expressed by $\frac{Q}{A}$ where Q is the charge on each plate then the potential difference between the two plates may be written

$$V = \left(\frac{d}{A\epsilon_0}\right)Q \qquad \textbf{Equation 5.1}$$

This equation tells us the intuitive result that for a given separation and plate area, the potential difference between two plates is directly proportional to the charge on the plates.

It would perhaps be sensible to now state the relation $V = kQ$ where k is a constant of proportionality with units of volts per coulomb and proceed from there. But historically the constant of proportionality is given by the reciprocal of k and we say that

$$Q = CV \qquad \textbf{Equation 5.2}$$

where the constant C is known as the capacitance of the parallel plate arrangement and is given by

$$C = \frac{\epsilon_0 A}{d} \qquad \textbf{Equation 5.3}$$

for the specific case of two parallel plates as described.

General case

In general of course, the case of two specific plates comes about only when specifically engineered, but the definition of capacitance is generalised for any conductor or pair of conductors according to equation 5.2 in the form

$$C = \frac{Q}{V}$$

If two conductors are present and one or both are charged, then there always exists a potential difference between them. The capacitance of the arrangement is the charge per potential difference. For symmetric arrangements this can be one absolute value for the whole setup and for asymmetric arrangements the capacitance will vary with position. This is sometimes referred to as mutual capacitance.

If one conductor is present and not connected to electrical ground then if it is charged it obtains a potential relative to a zero of potential at a large distance away. The capacitance of the conductor is defined as the charge stored per unit potential gained. In effect the capacitance is the charge that needs to be added to increase the potential by one volt. This is sometimes referred to as self capacitance.

Capacitors

Although we have computed the situation for the specific case of parallel plates, it turns out, both experimentally and theoretically that equation 5.2 is true for any two conductors separated by an insulator. Although equation 5.3 is specifically relevant for the parallel plate arrangement the relationship $Q \propto V$ holds true for any two conductors with an insulator in between, and the constant of proportionality is the capacitance of that particular setup. The following sections and problem sets provide examples and guidance on computing the capacitance of other arrangements of conductors.

Thusly, any arrangement of two conductors with an insulator in between - intentionally constructed or not – can be said to be a capacitor with a unique capacitance.

Capacitance

Capacitance is a scalar quantity with SI units of $\mathrm{CV^{-1}}$ or farads (F) and SI base units of $\mathrm{s^4A^2m^{-2}kg^{-1}}$. Using equation 5.3 an arrangement of two square conductors of side $1\ \mathrm{cm}$ held $1\ \mathrm{mm}$ apart will have a capacitance of $8.85\ \times\ 10^{-13}\ \mathrm{F} = 885\ \mathrm{fF}$. This means that if such a capacitor were fully charged by a $5\ \mathrm{V}$ battery the charge on each plate would be given by $Q = CV = 4.43\ \mathrm{pC}$ (and the related charge density $44.3\ \mathrm{nCm^{-2}}$).

One farad is defined as the capacitance of a capacitor with a potential difference of one volt between the plates when each is charged to one coulomb. It should be realised the one farad is an enormous, and usually completely impractical unit; diminishing SI prefixes are usually used.

Capacitance doesn't depend on the charge present

It is worth pointing out that the conductors do not have to have any charge on them to have a capacitance. The quantity is there anyway, defined by the geometry of the conductors, and will not vary irregardless of the amount of charge present on either conductor.

5.2 Some examples

The (self) capacitance of a conducting sphere

Consider a single conducting sphere of radius R carrying a charge Q. The potential on the sphere (section 4.2) is $\frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ and thus the charge per potential is simply given by $C_{sphere} = 4\pi\epsilon_0 R$.

So the capacitance of a typical laboratory van der Graaf generator of radius $10\ \mathrm{cm}$ is about $100\ \mathrm{pF}$ and the capacitance of the whole Earth is about $700\ \mathrm{\mu F}$.

The mutual capacitance of two concentric spheres

Consider an arrangement of two isolated conducting spheres with a common centre and radii R_1 and R_2 as shown in figure 5.2

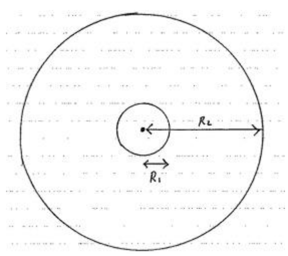

Figure 5.2 Cross section of two conducting concentric spheres

Consider the situation where the inner sphere is charged to $+Q$ and the outer sphere charged to $-Q$. Using the general equation for the potential difference between two points, the PD between the inner sphere and the outer sphere is given by

$$V_{21} = -\int_{1}^{2} \boldsymbol{E}.d\boldsymbol{r}$$

where E is the electric field between the two spheres.

From studying Gauss's law it is known that the only component of the field between the two spheres is radial and entirely due to the inner sphere having a magnitude of

$$\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

This then leads to the expression for the PD between the two plates as

$$V_{21} = -\frac{1}{4\pi\epsilon_0}Q\int_{R_1}^{R_2}\frac{dr}{r^2}$$

This has a general solution of

$$V_{21} = \frac{1}{4\pi\epsilon_0}Q\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

And so using the definition of capacitance from equation 5.2, the magnitude of the mutual capacitance between the two spheres is

$$C = \frac{4\pi\epsilon_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \equiv 4\pi\epsilon_0\frac{R_1R_2}{R_2 - R_1}$$

This is a neat result, but isn't one that most physicists would commit to memory; rather it is helpful it showing how two extremes corroborate earlier results and thus enhance understanding:

(i) A very large outer sphere

If the outer sphere is much larger than the inner sphere then mathematically $R_2 \gg R_1$ and thus the expression for the magnitude of the capacitance of the sphere just becomes $4\pi\epsilon_0R_1$ - just the result as the self capacitance of the inner sphere on its own.

When the self capacitance of a single object is considered, really it is as if an imaginary other object of infinite radius and opposite charge surrounds it; the mutual capacitance between these two objects is the same as the self capacitance of the finite-sized object on its own.

(ii) The spheres of similar sizes

If the two spheres are of similar sizes so the two spheres have a very small gap between then if the gap spacing is d, then $d = R_2 - R_1$ and capacitance is approximated by $4\pi\epsilon_0 \frac{R^2}{d}$ where R is the mean of the two radii. As $4\pi R^2 = A$ is the surface area of the midway point between the two spheres, the capacitance is approximately $\frac{\epsilon_0 A}{d}$ - a familiar result as it is the same as that of a parallel plate capacitor.

The mutual capacitance of a coaxial cable

A coaxial cable is a central wire with a larger cylindrical tube surrounding it. We consider the capacitance of an infinitely long coaxial cable consisting of a central straight infinitely long cylindrical conductor of radius R_1 surrounded by another infinitely long cylindrical conductor of larger radius R_2 such that they share a common axis. A schematic cross section is shown in figure 5.3

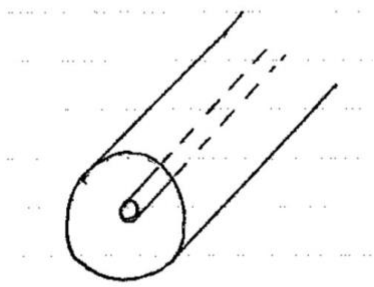

Figure 5.3 A cross section of a coaxial cable

Consider the situation in which the inner cylinder has a uniform charge distribution per unit length of λ Cm^{-1}. From section 3 on Gauss's law the electric field due to this infinite line of charge is

$$\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r}$$

where r is the distance from the axis of the cylinder, and is direct radially outwards.

Noting that only the inner cylinder can contribute to the field between the two cylinders, the potential difference between R_1 and R_2 is given by

$$V_{21} = -\int_1^2 \boldsymbol{E}.d\boldsymbol{r} = -2\lambda \int_{R_1}^{R_2} \frac{dr}{r} = -\frac{1}{4\pi\epsilon_0} 2\lambda \ln\frac{R_2}{R_1}$$

i.e. the PD is of magnitude $\frac{1}{4\pi\epsilon_0} 2\lambda \ln\frac{R_2}{R_1}$

So using the definition of capacitance as charge per potential, the capacitance per unit length of the coaxial cable is given by

$$C = \frac{2\pi\epsilon_0}{\ln\frac{R_2}{R_1}}$$

5.4 Energy stored in a capacitor

Capacitors are sometimes referred to as devices that are used to store electrical charge. This makes some sense, but isn't quite accurate. Consider two parallel plates again for example. Now consider, by whatever means, some electrons are taken from one of the plates and added to the other plate. The normal process of charging a capacitor is more indirect than this but regardless of the manner of charging the effect is the same: in a charged capacitor one of the plates has a surfeit of electrons and thus has a negative charge and the other plate has a dearth of electrons and a positive charge. The net charge is always zero and the number of charge carriers does not change. So to state that the capacitor stores charge is a little misleading without further context.

What a capacitor does store is energy in the form of electrostatic energy as an electric field between the plates. This stored energy – as with any type of stored energy – can be released to do work.

We now compute exactly how much energy is stored in a charged capacitor. Consider a parallel plate arrangement as shown in figure 5.4

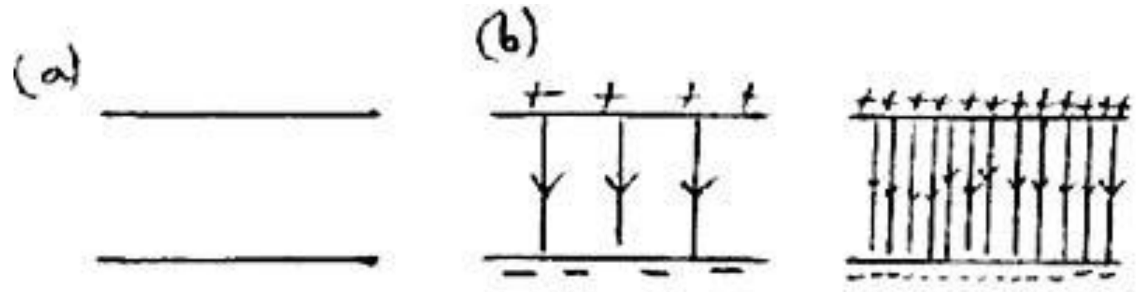

Figure 5.4 (a) an uncharged capacitor (b) a partially charged capacitor (c) a fully charged capacitor

Consider a simplified case in which a capacitor is charge by moving charged carriers from one plate to the other one by one until the capacitor is charged. Let us consider moving infinitesimally small positive charges, of magnitude δq from left hand plate to the right hand plate.

At first the electric field is zero so no work is done on moving the first charged carrier across.

The more charged carriers that have been moved across the plate the greater the electric field and the greater the potential difference between the plates. If, at any point, the magnitude of the PD between the plates is V then by the definition of PD the work done in moving a charge from one plate to another is

$$\delta W = V\delta q = \frac{q}{C}\delta q$$

where q is the charge that has already been transferred using the definition of capacitance.

Thus to charge a capacitor from a completely uncharged state until it has attained a total charge $\pm Q$ on each plate, the total work done is give by

$$W = \int_0^Q \frac{q}{C}.dq = \frac{Q^2}{2C}$$

This can also be written as $W = \frac{1}{2}QV = \frac{1}{2}CV^2$.

As this is the work done in charging the capacitor, it is thus the energy stored in the capacitor in the form of electrostatic potential energy, U_C. Thus

$$U_C = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$ **Equations 5.4**

So the electrical energy stored in a $500\ \mathrm{pF}$ capacitor charged to $5\ \mathrm{V}$ is about $600\ \mathrm{nJ}$.

Varying the distance between the plates

Now consider a parallel plate capacitor, charged to some fixed value where the distance between the plates may be changed. How will this affect what goes on between the plates?

First of the all the capacitance does change, as $C = \frac{\epsilon_0 A}{d}$ and thus the capacitance falls as the inverse of the separation.

The electric field strength will not change. As the field strength is given by $E = \frac{\sigma}{\epsilon_0}$, the charge density and plate area are constant thus the field strength remains the same regardless of the separation. Thus the field lines before and after would appear thus:

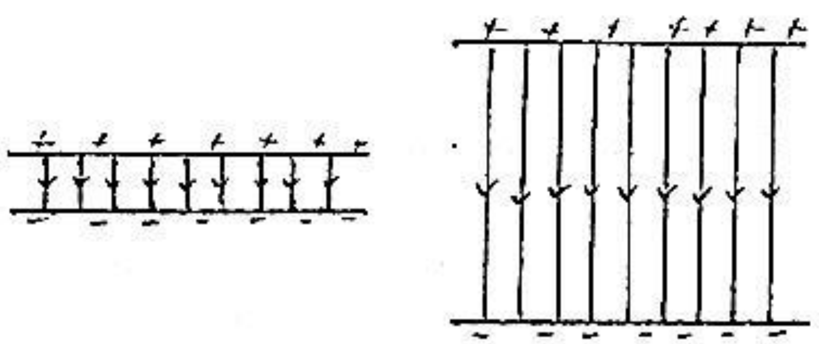

Figure 5.5 The electric field on a charged capacitor before and after separation

Similarly, the potential difference between the plates increases as $V = \left(\frac{d}{A\epsilon_0}\right)Q = \frac{Q}{C}$ the PD increases linearly with the separation.

Separating the plates must require some work to be put in. This is because the plates have unlike charge densities so feel an attractive force, thus separating them further must require work to be put in. Or looking at it another way, if the PD increases, there must be work done to increase it.

An easy way to calculate the work that must be put in is to recognise the energy stored is given by

$$U_C = \frac{Q^2}{2C} = \frac{Q^2 d}{2\epsilon_0 A}$$

so that also increases linearly with distance. The force between the plates might seem tricky to calculate at first but become easy when recognising that the relation between the force and the potential energy is

$$F = -\frac{dU}{dx}$$

in general so the force between the plates is simply

$$\frac{Q^2}{2\epsilon_0 A}$$

and is attractive i.e. it remains constant during the separation process. The work done is force multiplies by the distance moved.

Remember that these are approximate calculations that ignore the edge effects of the field; they are accurate only when the plate separation remains much smaller than the linear dimension of the plate size.

Energy density

The energy density of the electric field is defined as the electrostatic potential energy per unit volume. It is a scalar quantity with units of $\mathrm{Jm^{-3}}$. For the situation where edge effects of the field can be ignored, the electrostatic field and thus the energy stored between the plates of a parallel plate capacitor is entirely confined to this space, which is of volume Ad.

The energy density of the capacitor can be written in various ways, as can the energy, but the one that gives the most insight into the physics is by stating

$$U_C = \frac{1}{2}CV^2 = \frac{1}{2}.\frac{\epsilon_0 A}{d}.(Ed)^2 = \frac{1}{2}\epsilon_0 E^2.(Ad)$$

And thus the energy density of the capacitor is given by $\frac{1}{2}\epsilon_0 E^2$.

Though this has been shown explicitly for the uniform electric field between the plates of a capacitor this formula is much more general – it gives the local energy density for any electric field, viz.

$$u = \frac{1}{2}\epsilon_0 E^2 \qquad \textbf{Equation 5.5}$$

which is an equation which you will see many times throughout your study of electromagnetism.

5.5 Capacitors as one of the three main circuit components

Laboratory capacitors are not charged by simply shifting charges from one plate to the other. Instead they are connected by wires to a source of a constant potential difference (often a battery) which forms a complete electrical circuit with an effective resistance in series with the capacitor. The battery essentially pulls electrons away from one plate of the capacitor and pushes them towards the other. No charge moves between the plates, but they gradually build up equal and opposite charges.

At the start of the charging process the charging is easy, but as the charge builds up the process becomes more and more difficult, as the electrons on the positive plate become more reluctant to leave the and the electrons moving towards the negative plate become more reluctant to move towards this.

The charging process continues until the PD across the capacitor equals the PD supplied by the battery In fact the PD never quite equalises but becomes exponentially closer to that battery's PD with time. The current in the circuit starts at a maximum value given by the ratios PD of the battery to the effective resistance of the circuit and decreases exponentially with time.

If the battery is then disconnected the capacitor remains "charged" i.e. it keeps its electrostatically stored potential energy. It does this until the two connecting wires are then placed in series across a resistor of some kind.

In this case the capacitor supplies a PD across the resistor and a current flows as electrons from the negative plate flow towards the positive plate around the circuit until the PD on the capacitor falls to zero. As before this is an exponential decrease with time. The energy of the capacitor is dissipated by the resistor in the circuit. This dissipation of energy may be "wasted" as heat, or could be used for a useful purpose, such as making a light glow (like the flash of a camera) or for release of current at a useful time (like in an AC rectifier circuit).

Further details on the physics and especially the mathematics of the charging and discharging processes are the subject of electronics courses. Non-theory students are studying this as a compulsory unit in year 1; theory students may take electronics as an option later in the degree.

Capacitors are one of the most important three components in basic electronics along with the resistor and the inductor. In terms of energy:

- resistors dissipate electrical energy as heat
- capacitors store electrical energy in an electric field (and subsequently release it when required)
- inductors store electrical energy in a magnetic field (and subsequently release it when required).

The behaviour of all these components in direct current and alternating current circuits is essential knowledge for practitioners of practical physics. It does not add much to the theoretical analysis of electromagnetism however so that is as much as this lecture course will contain.

One more thing to remember about laboratory capacitors is that they have a polarity, i.e. they have a positive and negative terminal to work properly, and that the have a "voltage" rating which must not be exceeded. The reason for that is that there is never actually a vacuum between the plates, rather there is some kind of insulating material known as the dielectric. The dielectric materials tend to conduct in one direction but not the other and also have a breakdown "voltage" i.e. they conduct when the PD across them reaches a threshold value. The physics of dielectrics is discussed in the next section.

5.6 Dielectrics

Consider a parallel plate capacitor with a layer of material between the plates. If the material is nonconducting but each of the molecules in the material has a permanent dipole moment then a schematic diagram of the arrangement before and after the application of the field is shown in figure 5.5:

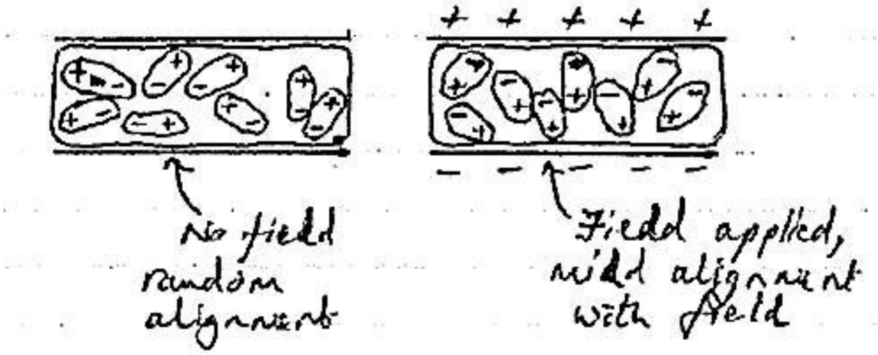

Figure 5.5 A capacitor filled with a substance with a dipole moment before and after the application of an electric field

Imagine that the capacitor is fully charged without the material layer being present, and then the material is quickly introduced. What happens to the electric field? When the material is placed in the field the dipoles start to align as shown in the figure. This means there is an accumulation of positive charge nearer to the negative plate and negative charge nearer the negative plate. The overall effect is to reduce the electric field in the material. Such materials are known as dielectrics.

It has been empirically shown that all nonconducting materials reduce the electric field by a constant factor – even those without a permanent dipole moment, as a field always induces a dipole moment in materials. If the field without the dielectric present has magnitude E_0 and the field with it present is E_D then the relation is given by

$$E_D = \frac{E_0}{\kappa} \qquad \textbf{Equation 5.6}$$

Correspondingly, the potential difference between the plates is given by

$$V_D = \frac{V_0}{\kappa}$$

and the capacitance of the arrangement is increased according to

$$C_D = \kappa C_0$$

The constant κ is known as the dielectric constant is a property of a material. It is a dimensionless scalar quantity and always has a value greater than unity. The dielectric constant of a vacuum is exactly 1, $\kappa_{air} \approx 1.0006$, $\kappa_{rubber} \approx 3$ and water, with its strong dipole moment has a particularly high value of about 20.

In addition, dielectric constants have a "dielectric strength" which is the breakdown field strength, i.e. the field strength that causes the dielectric to start conducting, which usually happens quickly and dramatically (like a lightning strike in air).

5.7 Displacement current

Consider a series circuits with a capacitor without a dielectric charging through a resistor as shown in the adapted circuit diagram shown below:

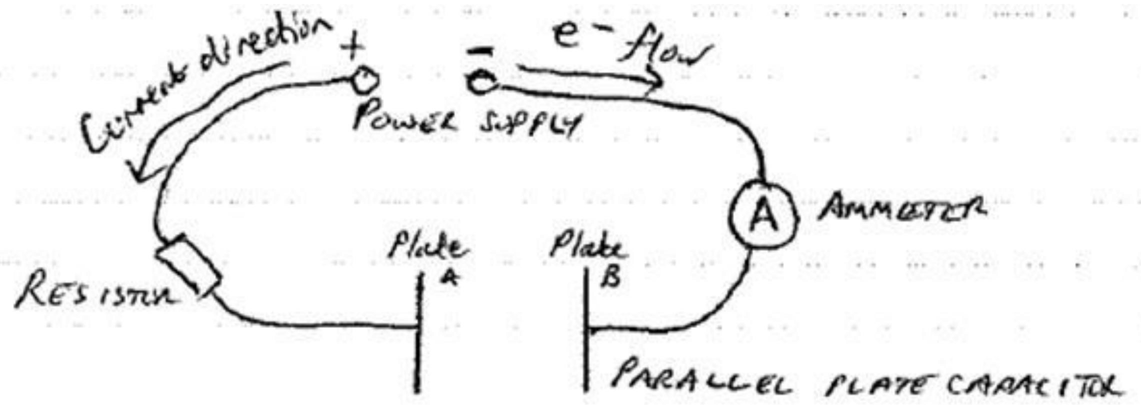

Figure 5.6 A charging capacitor

Consider the flow of charge in the circuit: apart from the power supply, the nature of which is unspecified, the flow of charge in the remainder of circuit consists of a flow of electrons. As this is a series circuit, the current at any time is a constant value at all points in the circuit i.e. the number of electrons passing a cross section of the circuit per unit time is a fixed value throughout.

Except between the plates of the capacitor.

Electrons build up on the right hand side of the capacitor, which accumulates a negative charge, and move away from the left hand side, which accumulates a positive charge. No electron move from positive to negative. No charge does. But there is an effect which is as if there is a current flow from one plate to the other: moving charges generate a magnetic field, and even though there is no charge flowing from between the plates of the capacitor there is still a magnetic field generated. This magnetic field is generated as a consequence of the changing electric field between the plates, and has a value that is the same as if there were a genuine flow of charge equal to the rate in the rest of the circuit. This is known as the displacement current and an appreciation of the presence of this quantity is a good place to leave off for the remainder of the course.