Electricity and Magnetism Part II:

Magnetism, Time-varying Fields and Maxwell's Equations

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Chapter 7

Static Magnetic Fields

7.1 Bar magnets

We all played with bar magnets as children. We found that not only did they stick to metal objects like 'fridge doors, they also interacted with each other. When placed close and parallel to each other, we found that in one orientation the magnets repelled each other, while in the other they were attracted. We can postulate that the ends of the magnets (or 'poles') are of two types and when a pair of parallel magnets are attracted to each other it is because the poles are either similar or dissimilar. With just two magnets we can't tell which it is. However, now introduce a third magnet. Let us say that two of the magnets, A and B, are oriented so that they are attracted to each other. We fix magnet A to the table and paint markers (say α and β) on each end of magnet B to identify each end. Now replace magnet B with magnet C and orient it so that it too is attracted to A. Since A is fixed, and the magnets are assumed identical, we know that C is oriented the same as was B, and we can paint the same makers (α and β) as we did for B. We can thus identify similar poles for B and C. Finally bring B and C parallel to each other and note that they are attracted to each other if the poles are opposite, and repelled from each other if they are the same. We can summarise our findings by saying that 'like poles repel, unlike poles attract'. A similar experiment can be done by simply suspending any magnet at its mid-point and seeing it align approximately along North-South (this is exactly what a compass needle does). Paint the north-pointing pole 'N' and the south-pointing pole 'S'. This experiment suggests, correctly, that the earth behaves like a bar magnet.

Whilst very easy to observe, describing the forces between magnets is *hard*. One problem is that the magnetic field **B** we shall introduce shortly is quite unlike the electric field **E** that we encountered earlier in the course. Whereas the action of an electric field on a charge is simple and intuitive, the action of a magnetic field on a charge is much less so. Even having described how (moving) charges are influenced by a magnetic field, we are still some way from describing quantitatively our childhood experiments with magnets. Our simple observations with bar magnets will guide us towards introducing the magnetic field. A quantitative description of how the field from one bar magnet influences another will have to wait until later. In fact, an important objective in this second half of the course is to try and demystify magnetic forces. Whilst a full treatment is beyond the scope of this course, I do hope at least to give a plausible sense of how magnetic fields arise, given our knowledge of electric fields. The laws describing how charged particles both generate and are influenced by magnetic fields will be discussed in detail. Finally, we will show how magnetic fields link to electric fields (and vice-versa), leading to Maxwell's equations.

In order to describe magnetic forces quantitatively, we need to describe the influence of a magnet throughout the surrounding space, i.e. we need to introduce a magnetic *field*, **B**. It is easy to visualise the field lines associated with a bar magnet by placing the magnet on a sheet of paper and sprinkling with iron filings (see Fig. 7.1). Each filing acts as a little bar magnet which orients itself as best it can in response to 'like (unlike) poles repel (attract)' rule. The field lines apparently run between the poles. In order to determine in which direction the field lines run (i.e. North to South or vice versa) we need to look at the force experienced by a charge moving through a magnetic field.

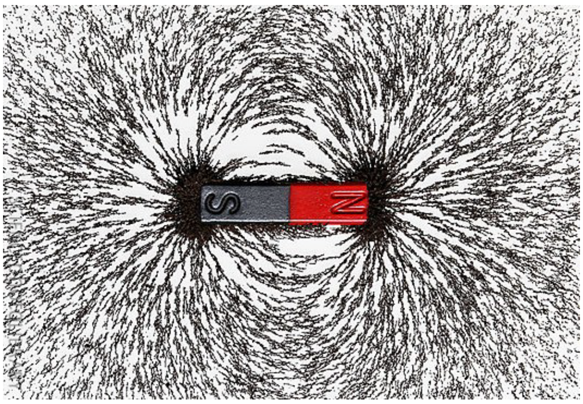

Fig. 7.1 Visualising the magnetic field near a bar magnet with iron filings.

7.2 Movement of Charges in a Magnetic Field - the Lorentz Force Law

We find experimentally that when charges move through a magnetic field they feel a force. There is a nice YouTube video that illustrates this:

Magnetic field and cathode rays

If you look carefully, you will notice that the magnet is actually 'horse-shoe' shaped, and it is the ends that are brought up to the particle beam which moves from left to right. The set-up is shown schematically in Fig. 7.2.

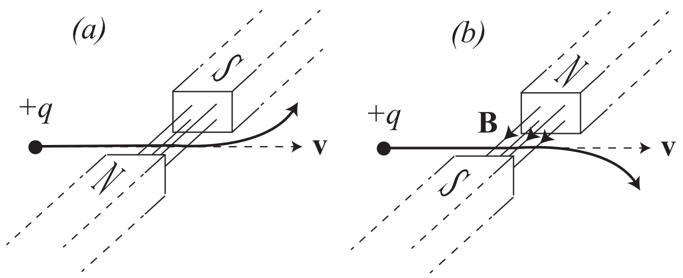

Fig. 7.2 (a) Movement of charge in a magnetic field; (b) As for (a) but now with the direction of the magnetic field (**B**) reversed.

The magnetic field runs between the poles, and the charged particles respond to the field in a rather odd way. The particles are moving horizontally in the plane of Fig. 7.2 whilst the magnetic field is also horizonal, but out of the plane of the figure. The charged particles are deviated *vertically*, and when the magnet is turned through 180°, the beam of charged particles is deflected the opposite way.

Let's try to describe this mathematically. When a charged particle is deviated in a magnetic field its deviation is in a direction perpendicular to both the particle's velocity vector **v** and the line of the magnetic field. We still haven't fixed a direction for the magnetic field, but will do so shortly. Let us assume that the particles are *positively* charged (for cathode rays the particles are actually electrons which are negatively charged, but it doesn't matter provided we are consistent). Now let us suppose that the poles are oriented such that the (positively charged) particles are deviated *downwards*. Experimentally this is found to be the configuration of Fig. 7.2(b). The direction of the deviation, or the force due to the magnetic field, is given by $\mathbf{v} \times \mathbf{B}$, provided we agree that *the direction of the magnetic field,* **B***, runs from the North pole to the South pole.* This direction is determined by the right-handed convention of the cross-product. If we chose a left-handed cross-product then the field direction would have to be defined as running South-to-North, and no error would result.

Experimentally, the magnitude of the force felt by the particle is proportional to its charge and its speed. Thus the magnetic force experienced by the particle is

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}. \tag{7.1}$$

This equation gets the correct direction for the force for the given convention of the direction of **B**. Since $\mathbf{F}_m$, q and $\mathbf{v}$ are either known or measurable, Eq. (7.1) also defines the *magnitude* of **B**, the magnetic field strength.[1] This equation sets the units of the magnetic field as $\mathrm{kgC^{-1}s^{-1}}$, or tesla (T). With the direction of the magnetic field now set, we can add arrows to the field lines around a magnet as shown in Fig. 7.3.

One important point should be emphasised about magnetic forces:

Magnetic forces do no work

To see this we write Eq. (7.1) as

$$\mathbf{F}_m = q\frac{d\mathbf{r}}{dt} \times \mathbf{B}. \tag{7.2}$$

Now recall that the work done by a force in moving a particle through a displacement $d\mathbf{r}$ is $dW = \mathbf{F} \cdot d\mathbf{r}$, and we see from Eq. (7.2) that in this case $\mathbf{F}_m \cdot d\mathbf{r} = 0$. From the work-energy theorem in mechanics we conclude that magnetic forces acting on a particle cannot increase its kinetic energy.

If there is an electric field **E** present in addition to the magnetic field, then the total electromagnetic force experienced by the particle is described by the vector equation

$$\mathbf{F}_{em} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right), \tag{7.3}$$

which is called the **Lorentz force law**.

You might ask yourself why we combine the magnetic force with the electric force in this way. Why not include (say) gravitational forces as well? The answer is that electric and magnetic fields are ultimately both part of the same

[1] Actually, for reasons that will become clearer later, the field **B** is often referred to as the **magnetic flux density**.

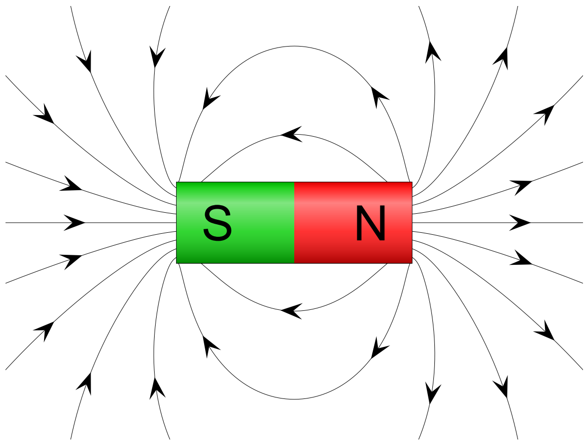

Fig. 7.3 Magnetic field lines around a bar magnet

entity - the *electromagnetic field*. Showing exactly how this works is beyond the scope of this course, but we can get a flavour of what's going on by imagining we are sitting watching a static charge in the Laboratory reference frame so that there will be the familiar radial electric field lines $\mathbf{E}$. How will this field be observed by a colleague who is in uniform relative motion? In the moving frame the charge is moving, and we know that a moving charge is effectively a current. We will see below that currents are sources of magnetic fields. So in the moving frame the charge is a source of both a magnetic field $\mathbf{B}'$ and an electric field $\mathbf{E}'$. Roughly speaking, in the moving frame components of the $\mathbf{E}$ field that lie in the direction of motion are contracted leading to a bunching of the field lines *transverse* to the motion as depicted in Fig. 7.4. This results in an intensification of the field transverse to the motion that we identify as being the action of a magnetic field. The exact description of how electromagnetic fields in the Lab frame $(\mathbf{E}, \mathbf{B})$ transform to the fields in the moving frame $(\mathbf{E}, \mathbf{B})$ requires advanced techniques (covered in the Advanced Classical Physics course), but you hopefully get the idea from this that Special Relativity plays a fundamental role in understanding the unity between electric and magnetic fields.

7.2.1 ***The Hall effect***

Just as charges curve when moving in vacuum in the presence of a transverse magnetic field, so too do free charges within conducting materials. The

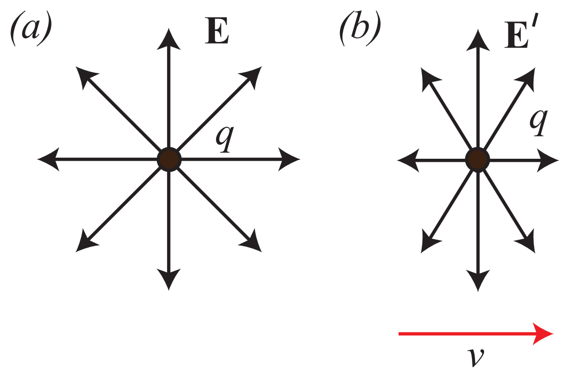

Fig. 7.4 Two views of the electric field lines from a positive charge: (a) Lab frame (b) in a frame moving to the right with speed v.

resulting charge separation can set up an electric field, the sign of which depends on the sign of the conduction charges. This is the basis of the so-called *Hall effect*, discovered by Edwin Hall in 1879 when he was an undergraduate.

Consider the conducting strip of height d shown in Fig. 7.5(a). Negative charge carriers ($-q$) move with velocity $\mathbf{v}$ pointing in the $-x$ direction, yielding a current density in the strip $\mathbf{J}$ pointing along $+x$. A magnetic field $\mathbf{B}$ pointing in the $+y$ direction causes the charges to drift *upwards* under the influence of the Lorentz force $-q\mathbf{v} \times \mathbf{B}$. Negative charge accumulates on the top edge of the strip, leaving an excess of equal positive charge on the lower strip. The resultant *Hall field* $\mathbf{E}_H$ points along $+z$. Now consider Fig. 7.5(b) where the charge carriers are positive ($+q$). A similar analysis shows that now the upper edge becomes positively charged, whilst the lower edge becomes negatively charged. The Hall field $\mathbf{E}_H$ now points along $-z$. In equilibrium the Lorentz force on any conducting charge is balanced by the force qE_H. In the situation of Fig. 7.5(b) the equilibrium condition is $qvB+qE_H = 0$ showing that $E_H = -vB$. For 7.5(a) $E_H = +vB$.

The Hall field (or rather the Hall voltage,[2] $V_H = E_H d$), together with the magnitude of the applied magnetic field B and current $I = JA$ (where A is the cross-sectional area of the strip) are all experimentally known quantities. We can use these to determine the sign and number density n, of the charge carriers in the sample. For the situation in Fig. 7.5(b) we have, since $J = nqv$ that

[2] Shh...don't tell Dr Tymms I said 'voltage' when I should have said 'potential difference'!

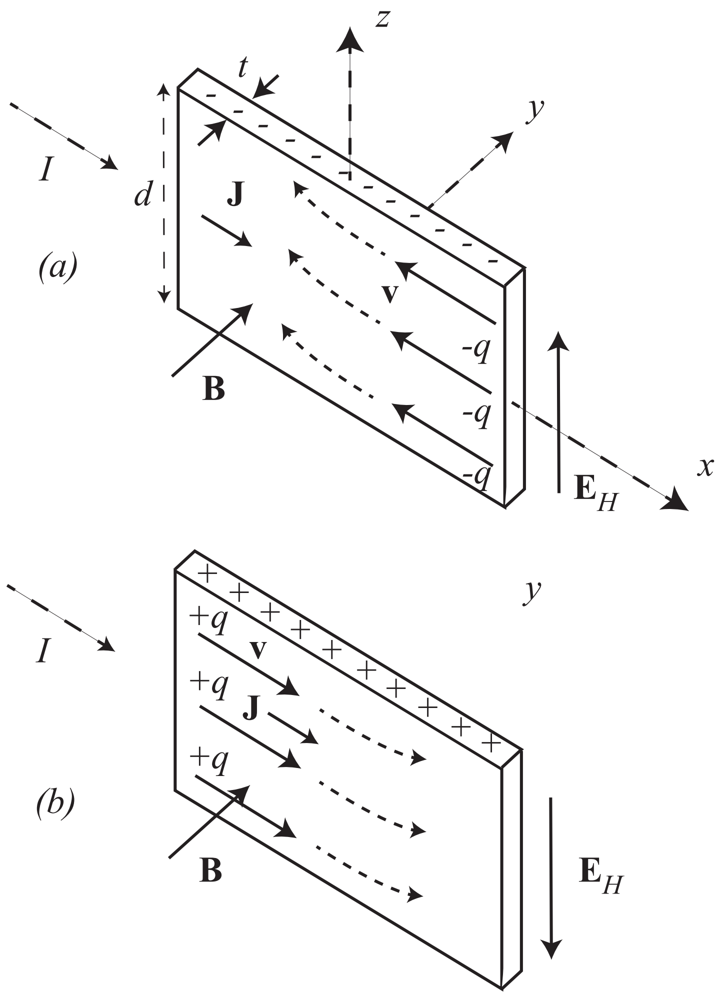

Fig. 7.5 The Hall effect: (a) negative conduction charges, (b) positive conduction charges.

$$E_H = \frac{V_H}{d} = -vB = -\frac{JB}{nq} = -\frac{IB}{nqA}. \tag{7.4}$$

From this we see that

$$nq = -\frac{IBd}{V_H A} = -\frac{IB}{V_H t}, \tag{7.5}$$

where t is the thickness of the strip. In Eq. (7.5) nq is a positive quantity, since $V_H < 0$. For Fig. 7.5(a) we would have $nq < 0$, since then $V_H > 0$. Hence the sign of the charge carriers, and their number density, can be experimentally determined. For metals the charge carriers are negative electrons. But for semiconductors the charge carriers can be positive 'holes', where a hole, the *absence* of a negative charge, is equivalent to a positive charge.

The Hall effect can also be used to measure magnetic fields if the properties of the sample (n and the sign of the conducting charges) are known. A *Hall probe* can be used to determine the size of the transverse magnetic field via

$$B = \frac{nqV_H A}{Id} = \frac{nqV_H t}{I}. \tag{7.6}$$

7.2.2 *Circular motion of charged particles in a uniform magnetic field*

When charged particles enter a magnetic field they travel along curved paths. This results from the strange *transverse* nature of the Lorentz force law; the magnetic force is perpendicular to both the direction in which the charge is moving, *and* the direction of the magnetic field. For non-uniform magnetic fields (or when both a magnetic field and electric field are present) the motion of a charged particle can be quite complicated; we will just analyse the motion of a charged particle in a uniform magnetic field as shown in Fig. 7.6. Even solving this problem rigorously is quite hard; fortunately we can get all of the physics much more slickly using a heuristic[3] argument that we will give after the formal solution. Newton's second law for a charged particle of mass m is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}. \tag{7.7}$$

For definiteness, let us say that $\mathbf{B}$ is in the $+z$ direction, i.e. $\mathbf{B} = B\mathbf{k}$. The z-component of Eq. (7.7) is easy to solve, since for that direction we have $m dv_z/dt = 0$, and $v_z = const$. The equations for the other two directions are

$$m\frac{dv_x}{dt} = qv_y B, \quad m\frac{dv_y}{dt} = -qv_x B, \tag{7.8}$$

which may be combined (by differentiation) as

[3] A fancy word for 'handwaving'.

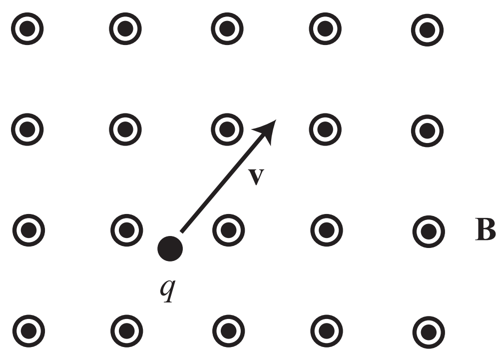

Fig. 7.6 Charged particle moving in a region of uniform magnetic field.

$$\frac{d^2 v_{x,y}}{dt^2} + \omega_c^2 v_{x,y} = 0, \tag{7.9}$$

where

$$\omega_c = \frac{|q|B}{m} \tag{7.10}$$

is known as the *cyclotron frequency* (we put $|q|$ to ensure that the frequency is positive[4]). In Eq. (7.9) the notation $v_{x,y}$ means we can choose either v_x or v_y, so really the single equation represents two equations. As I hope you can see, Eqs. (7.9) are simple harmonic motion equations at the cyclotron frequency, ω_c. Now you may say, so that's easy, I can just use the general solution for SHM from the Vibrations and Waves course to write down the general solutions for $v_x(t)$ and $v_y(t)$ as

$$v_x(t) = A\cos(\omega_c t) + B\sin(\omega_c t), \tag{7.11}$$

$$v_y(t) = C\cos(\omega_c t) + D\sin(\omega_c t), \tag{7.12}$$

where A, B, C and D are arbitrary constants. But it's not so simple. The general solution to the pair of differential equations (7.8) has only *two* arbitrary constants, whereas Eqs. (7.11) and (7.12) has four. The constants A, B, C

[4]If the sign of q is retained, the sign of ω_c in the solution determines the *sense* of rotation. However, it's usually easier to figure out the sense of rotation by applying the Lorentz force law to the particle at a given instant.

and D are not independent, but are fixed by the requirement that in addition to solving the pair of Eqs. (7.9), the solutions must also satisfy the original Eqs. (7.8). In fact, the general solution for $v_x = dx/dt$ and $v_y = dy/dt$ is

$$\begin{aligned} v_x(t) &= \frac{dx}{dt} = v_x(0)\cos\omega_c t + v_y(0)\sin\omega_c t, \\ v_y(t) &= \frac{dy}{dt} = -v_x(0)\sin\omega_c t + v_y(0)\cos\omega_c t, \end{aligned} \tag{7.13}$$

where the two arbitrary constants are expressed as $v_{x,y}(0)$, the velocity components at $t = 0$. If you fancy a challenge you may care to derive these equations for yourself. A very neat way is to introduce the complex variable $\tilde{v} = v_x + iv_y$, and show that Eqs. (7.8) may be written as $d\tilde{v}/dt = -i\omega_c\tilde{v}$. Then, if you fancy a *real* challenge, see if you can solve $m\dot{\mathbf{v}} = \mathbf{F}_{em}$, with $\mathbf{F}_{em}$ given by Eq. (7.3) for the case of constant and uniform $\mathbf{E}$ and $\mathbf{B}$. It's tricky, but is doable using techniques within your range.

Integrating Eqs. (7.13) to obtain the particle trajectory is relatively straightforward. The result is

$$\begin{aligned} x(t) &= x(0) + \omega_c^{-1}\left[v_x(0)\sin\omega_c t + v_y(0)(1-\cos\omega_c t)\right], \\ y(t) &= y(0) + \omega_c^{-1}\left[v_y(0)\sin\omega_c t - v_x(0)(1-\cos\omega_c t)\right]. \end{aligned} \tag{7.14}$$

The last two equations can be combined to form

$$\left[x - x(0) - v_y(0)/\omega_c\right]^2 + [y - y(0) + v_x(0)/\omega_c]^2 = v^2/\omega_c^2, \tag{7.15}$$

which is a circle of radius $R = v/\omega_c = mv/|q|B$ centred at $(x(0) + v_y(0)/\omega_c, y(0) - v_x(0)/\omega_c)$.

Fortunately there is a much quicker way to deduce that the motion is circular in the x-y plane with a radius of mv/qB. We know from Eq. (7.7) that since $\mathbf{v}_\perp = v_x\mathbf{i} + v_y\mathbf{j}$ is orthogonal to $\mathbf{B} = B\mathbf{k}$, the force is of magnitude qvB and is *always* directed perpendicular to $\mathbf{v}_\perp$. This is just what happens in circular motion: the force is always perpendicular to the velocity, directed towards the centre of the circle. The magnitude of the force can be equated to the centripetal force required to keep the particle moving in a circle of radius R:

$$|q|vB = \frac{mv^2}{R}, \quad \text{or } R = \frac{mv}{|q|B}. \tag{7.16}$$

The period of the motion is just

$$T_c = \frac{2\pi}{\omega_c} = \frac{2\pi m}{|q|B}. \tag{7.17}$$

Notice that the period is independent of the speed v; the faster the particle moves the larger the radius, but the frequency and the period stay the same.

Once we remember that $v_z = const.$, we see that the resultant motion of a charged particle in a uniform magnetic field is a *helix* - see Fig. 7.7

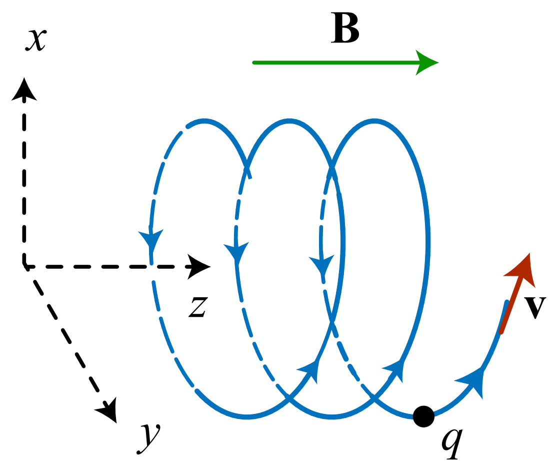

Fig. 7.7 Helical motion of a charged particle moving in a region of uniform magnetic field.

The motion of a charged particle in a magnetic field impacts several areas of physics. Detailed magnetic field configurations are used to confine the particle trajectories to fixed regions of space in attempts to confine hot ($\sim 10^6$ K) plasmas for nuclear fusion experiments. These are so-called *magnetic bottles*. Professor Steve Cowley FRS and colleagues in this department are very much involved with that effort. The Earth's non-uniform magnetic field traps particles streaming from the sun in large regions called the *Van Allen radiation belts*. Ionisation collisions of these particles at the poles cause the aurora borealis. Spiral tracks of charged particles in bubble chambers in high energy physics experiments are analysed to determine the mass and charge of the particles. In the following we will look at a couple of other simple examples.

7.2.3 ***Mass Spectrometry***

A mass spectrometer uses the fact that the radius of the orbit of an ion in a uniform magnetic field depends on the mass. Different isotopes of the same element will have different radii, enabling the mass to be determined. Figure 7.8 shows ions of charge q and mass m being accelerated through a potential

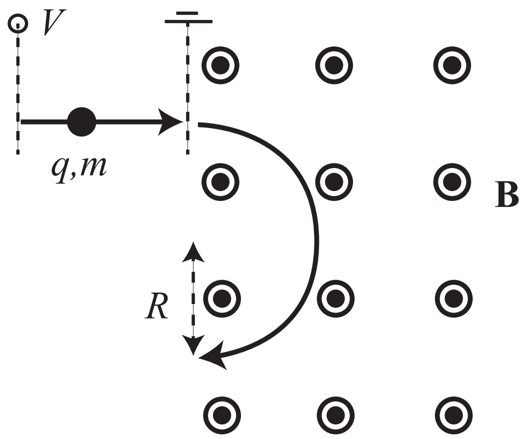

Fig. 7.8 Mass spectrometer.

V before entering the region of uniform magnetic field. Non-relativistically (i.e. $v \ll c$), the kinetic energy of the ion on entering the **B**-field region is given by

$$\frac{1}{2}mv^2 = qV, \quad \Rightarrow \quad v = \left(\frac{2|q|V}{m}\right)^{1/2}. \tag{7.18}$$

Using this in the expression for the radius (cf. Eq. (7.16)) shows that

$$R = \frac{mv}{|q|B} = \frac{m}{|q|B}\left(\frac{2|q|V}{m}\right)^{1/2} = \left(\frac{2mV}{|q|B^2}\right)^{1/2}, \tag{7.19}$$

showing that $R \propto m^{1/2}$. The radius is measured by observing the position of the ion on travelling through a semicircle as shown. The same technique can be used for isotope separation.

7.2.4 *Cyclotron*

The cyclotron is a type of charged particle accelerator that utilises the fact that non-relativistically the frequency ω_c of the circular motion is independent of the speed. It is why the cyclotron frequency was so-named. Figure 7.9 shows two D-shaped electodes across which is applied an alternating voltage $V(t) = V_0 \cos \omega_c t$. The uniform magnetic field perpendicular to the electrodes causes an injected charged particle to move in a circular path (the electrodes are hollow). On reaching the gap, the charged particle is accelerated by the voltage and acquires kinetic energy qV_0. Then, half a cycle later, the particle enters the gap from the other electrode. At this time the voltage has reversed, so that the particle is accelerated again, and picks up a further qV_0 of kinetic energy. Each time the particle crosses the gap it gains kinetic energy and circles at a larger radius, so that starting from near the middle of the electrodes it spirals out to the edge, always picking up energy synchronous with the alternating voltage. The technique, invented in 1931 by Lawrence and Livermore at the University of California, could accelerate particles up to energies of about 4 MeV[5] using a cyclotron about 70 cm in size. In 1931 Lawrence won the Nobel prize for this work. The largest cyclotron today is at the RIKEN laboratory in Japan. Its electrodes are about 19m in diameter, and with a maximum field of 3.8T, it has accelerated Uranium ions up to 81GeV.

A limitation of the cyclotron is that when relativistic effects are accounted for, the period of the orbiting ion depends on velocity, so that $\omega_c(v)$. Other methods then become necessary, such as the synchrotron. The Large Hadron Collider at CERN is an exmple of a synchrotron, and is the largest particle accelerator in the world, capable of reaching particle energies of 7 TeV.

7.2.5 *Discussion questions*

(1) At any point in space, the electric field is defined to be in the direction of the electric force on a positively charged particle at that point. Why don't we similarly define the magnetic field **B** to be in the direction of the magnetic force on a moving, positively charged particle?
(2) If a magnetic force does no work on a charged particle, how can it have any effect on the particle's motion?
(3) In a Hall-effect experiment, is it possible that no potential difference will be observed? Under what circumstances might this happen?

[5] 1 MeV = 1.6×10^{-13} J.

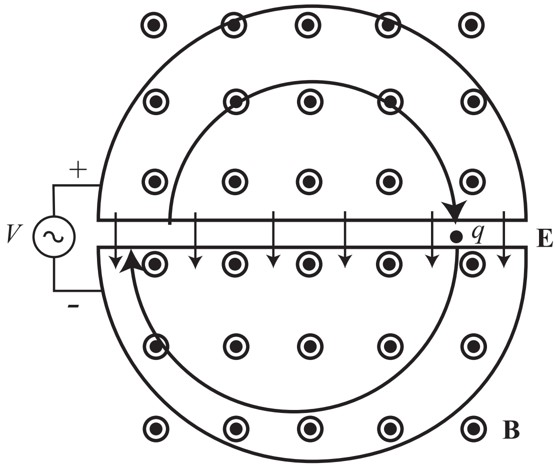

Fig. 7.9 Cyclotron.

7.2.6 *Numerical Exercises*

Data for these questions: Atomic number density of copper = 8.5×10^{28} m^{-3}, resistivity of copper = $1.7 \times 10^{-8}\Omega$m.

(1) A piece of copper wire radius 0.5 mm carries a uniformly distributed current of 1 A. Find the electron drift speed in the wire. (You may assume that each copper atom provides a single free electron.)
[For comparison, the thermal speed of the electrons at 20^{o}C is about 10^5ms^{-1}.]
(2) If the wire in (1) above is 100 m long, calculate the energy dissipated in the wire in one minute.
(3) A straight copper bar with *rectangular* cross section is aligned along the z-axis and carries a current of 3A in this direction. A magnetic field **B** of magnitude 1 T is applied in the x-direction, perpendicular to two sides of the bar that are separated by 1.5 mm. In what direction does the Hall effect voltage appear, and what is its magnitude?

7.2.7 *Problems*

(1) Estimate the approximate maximum deflection of the electron beam near the centre of a TV screen due to the earth's 5.0×10^{-5} T field. Assume the

screen is 20 cm from the electron gun where the electrons are accelerated by (a) 2.0 kV, or (b) by 30 kV. Note that in the old cathode ray-based colour TV sets, the beam must be directed accurately to within less than 1 mm in order to strike the correct colour-producing phosphor. Did the old TV sets need to take any special precautions to offset this effect?

(2) Explain why the following situation is impossible. Fig. 7.10 shows an experimental technique for altering the direction of travel for a charged particle. A particle of charge q = 1.0 μC and mass m = 2.00 $\times$ 10^{-13} kg enters the bottom of the region of uniform magnetic field at speed v = 2.00$\times$$10^5$ ms^{-1}, with a velocity vector perpendicular to the field lines. The magnetic force on the particle causes its direction of travel to change so that it leaves the region of the magnetic field at the top traveling at an angle from its original direction. The magnetic field has magnitude B = 0.400 T and is directed out of the page. The length h of the magnetic field region is 0.110 m. An experimenter performs the technique and measures the angle θ at which the particles exit the top of the field. It is found that the angle of deviation is exactly as predicted.

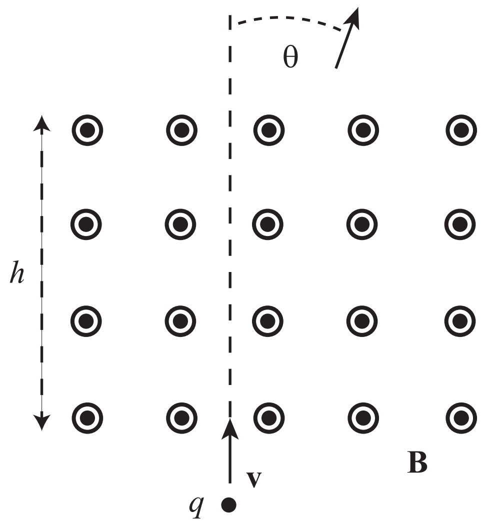

Fig. 7.10 A possible experimental method for deviating charged particles.

7.3 Magnetic Forces on Currents

7.3.1 *Force on a current-carrying wire*

Currents are just moving charges, so wires carrying currents must feel a Lorentz force in a magnetic field. Let's calculate this force. The force on each charge q comprising the current is $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. Multiplying this by n, the number density of charge carriers, we find that the force *per unit volume* on the wire is

$$\mathbf{F}_V = \mathbf{F} \textit{ per unit volume } = nq\mathbf{v} \times \mathbf{B} = \mathbf{J} \times \mathbf{B}. \tag{7.19}$$

By multiplying by (or integrating over) the cross-sectional area of the wire, and assuming $\mathbf{B}$ is constant over the area, we can now use Eq. (6.3) (or Eq. (6.4), for non-uniform current densities) to obtain the force *per unit length* on the wire as

$$\mathbf{F}_\ell = \mathbf{F} \textit{ per unit length } = I\hat{\mathbf{n}} \times \mathbf{B}, \tag{7.20}$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of current the current, I. Finally, the force on a length ℓ of the wire is given by

$$\mathbf{F} = I\hat{\mathbf{n}} \times \mathbf{B}\ell. \tag{7.21}$$

7.3.2 *Torque on a current loop - magnetic dipole moment*

Actually, single sections of wire don't occur in practice. We always deal with current *loops*, so it is relevant to consider the effect of a magnetic field on a loop. Now provided the magnetic field is uniform, the net force on a loop is zero. To see this consider the loop carrying a current I in Fig. 7.11. If we isolate a small section of the loop of length $d\ell$, then by Eq. (7.21) we see that the Lorentz force felt by that section is

$$d\mathbf{F} = I\hat{\mathbf{n}}d\ell \times \mathbf{B} = I d\boldsymbol{\ell} \times \mathbf{B} \ .$$

The total force on the loop is therefore

$$\mathbf{F} = I \oint_C d\boldsymbol{\ell} \times \mathbf{B} \ ,$$

which of course vanishes as $\oint_C d\boldsymbol{\ell} = 0$. So it looks like examining the forces on current loops will not be very interesting. However, although the net force on a current loop vanishes, there can be a net torque, and the torques that (the equivalent of) little current loops in magnetic materials feel turns out to be rather important in understanding the magnetic properties of materials.

Consider the rigid rectangular current loop in a uniform magnetic field, as shown in Fig. 7.11. The loop is constrained to rotate about the axis AC. The plane of the loop makes an angle θ to the vertical when the **B**-field is horizontal as shown. The sides of length a feel a magnetic force directed away from the centre of the loop and parallel to AC. The sides of length b on the other hand, each feel a force of magnitude BIb directed as shown. Together these two forces exert a couple $Fa\sin\theta = BIab\sin\theta$ on the loop. We can express this formula nicely by associating a vector **A** with the loop, whose magnitude is the area $A = ab$ and directed according to the right-hand rule applied to the current in the loop. Thus the torque $\mathbf{\Gamma}$ on the loop is given by $\mathbf{\Gamma} = I\mathbf{A}\times\mathbf{B}$. Now the vector quantity $I\mathbf{A}$ occurs frequently, and so is given a special symbol $\boldsymbol{\mu}$ and the name *magnetic dipole moment*[6]. Therefore

$$\mathbf{\Gamma} = \boldsymbol{\mu}\times\mathbf{B}. \tag{7.22}$$

You can perhaps see why $\boldsymbol{\mu}$ is given the name magnetic dipole moment. Recall that an electrostatic dipole **p** in the presence of an electrostatic field feels a torque $\mathbf{p}\times\mathbf{E}$ (cf. Eq. (2.6)), so evidently the magnetic dipole moment $\boldsymbol{\mu}$ plays the same role in magnetism as is played by the electric dipole moment **p** in electricity. Sometimes we abbreviate to just *magnetic moment*.

By calculating the work done against the magnetic torque in Eq. (7.22) we can assign a potential energy U with the configuration shown in Fig. 7.11. In rotating the loop from $\theta = 0$ to $\theta = \Theta$ the work done against the torque is

$$W = \int_0^\Theta |\mathbf{\Gamma}|\,d\theta = \int_0^\Theta |\boldsymbol{\mu}\times\mathbf{B}|\,d\theta = \int_0^\Theta |\boldsymbol{\mu}|\,|\mathbf{B}|\sin\theta d\theta$$

$$= -\,|\boldsymbol{\mu}|\,|\mathbf{B}|\cos\theta|_0^\Theta = -\,\boldsymbol{\mu}\cdot\mathbf{B}|_0^\Theta = \mu B(1-\cos\Theta)\ . \tag{7.23}$$

The torque is conservative in that it depends on θ, but not on $d\theta/dt$, and we know from mechanics that in such situations we can associate the work done with a potential function $U(\theta)$

$$U(\theta) = -\mu B\cos\theta\ , \tag{7.24}$$

[6]Sometimes the symbol **m** is used for the magnetic dipole moment.

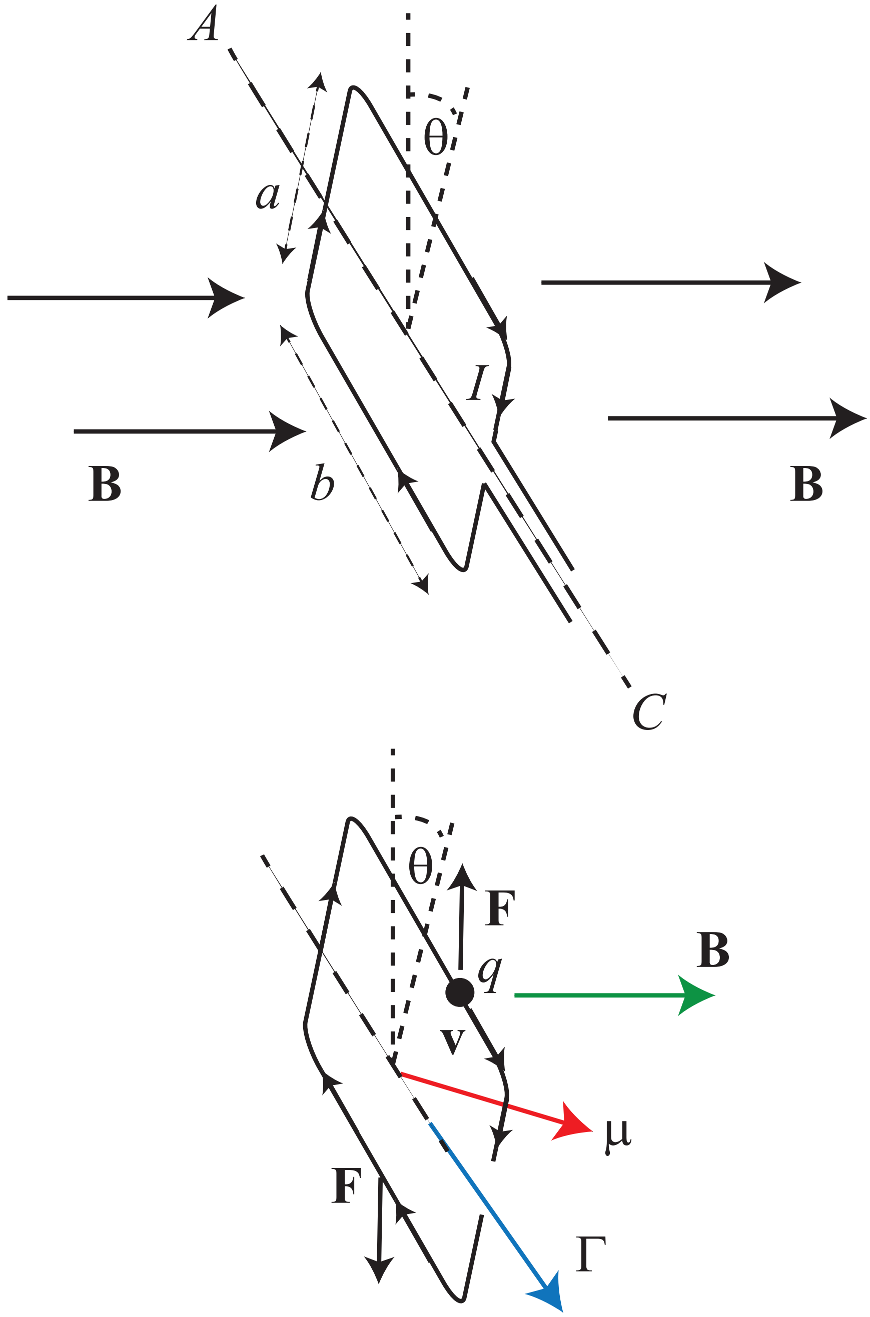

Fig. 7.11 Torque on a current loop

where the zero of potential is identified as the orientation $\theta = \pi/2$. We then note that the orientation $\theta = 0$ is a potential minimum, which is therefore a *stable* equilibrium. In the absence of friction a magnetic dipole in a magnetic field will oscillate around $\theta = 0$. The orientation at $\theta = \pi$ where the dipole is anti-aligned with the field is a potential maximum, and therefore an *unstable* equilibrium. The torque points in the z-direction and is given by

$$\Gamma = -\frac{dU}{d\theta} = -\mu B \sin\theta \, . \tag{7.25}$$

This gives the right magnitude ($= \mu B \sin\theta$) and the sign shows that the torque is always restoring, i.e. it acts so as to drive the system back towards the stable equilibrium orientation at $\theta = 0$. The difference in potential energy between the configurations when the dipole moment $\boldsymbol{\mu}$ is aligned ($\theta = 0$) and anti-aligned ($\theta = \pi$) with the magnetic field $\mathbf{B}$ is just $2\,|\boldsymbol{\mu}|\,B$.

Although we have worked this out for a rectangular loop, it is not hard to show that Eqs. (7.22)-(7.24) are valid for loops of any shape.

7.3.3 *The Electric Motor*

As we saw above the torque experienced by a current loop in a magnetic field causes the loop to oscillate about the equilibrium orientation ($\theta = 0$). This is true even for very large angular amplitudes such as when the magnetic moment starts almost anti-aligned ($\theta \approx \pi$) with the $\mathbf{B}$-field. The torque is always restoring, or in other words it changes sign as the moment passes through equilibrium. Now if some clever way could be devised to *reverse* the current in the loop just when it passes through equilibrium, then the torque would *not* change sign and the current loop would rotate even faster. Having the current switch every time the loop passes through equilibrium (both at $\theta = 0$ and $\theta = \pi$) will cause the loop's angular speed to increase until frictional forces in the bearings oppose the angular acceleration. This is the basis for a D.C. electric motor as shown in Fig. 7.12. Current is driven through a loop that is free to rotate about its axis between the poles of a magnet. The clever thing about a DC electric motor[7] is the so-called *commutator* which consists of semi-circular contacts that connect to the circuit via brushes. As the loop traverses the vertical the sign of the current reverses, and the loop speeds up as discussed above. A stronger magnetic field can often be obtained by using an electro-magnet (which we will describe later), rather than a permanent magnet. Instead of a commutator, an alternating current could be used instead to create an A.C. motor. In that case the final angular speed of the motor is set by the frequency of the alternating current. Both D.C. and A.C. motors are a means of converting electrical energy into mechanical energy. If you find

[7] I don't know who thought of it first, but it's an awfully clever idea.

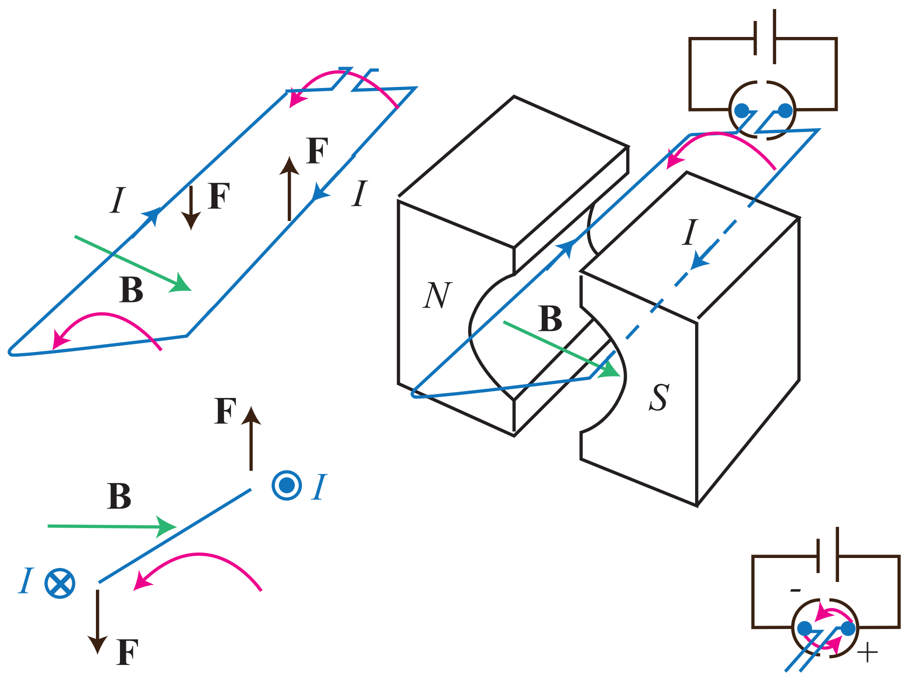

Fig. 7.12 A DC electric motor.

my artwork in Fig. 7.12 and explanation too confusing, then I recommend the following YouTube video:

YouTube: DC motor principle

7.3.4 *Magnetic Materials*

We have now learnt enough to explain, qualitatively, some features of magnetic materials, such as why a bar magnet attracts a nail, and how a paper clip can become magnetised in the presence of a bar magnet. We will be best placed to do this, however, after discussing the microscopic origin of magnetism and magnetic materials.

Consider an electron orbiting a nucleus as shown in Fig. 7.13. If we assume a classical circular orbit (of radius a) then it is straightforward to see that the atom must have a magnetic moment. The current is just the charge passing a given point on the orbit every second. The period of each orbit is just $2\pi a/v$, so the charge passing every second is just $ev/2\pi a$, or in terms of the angular momentum $L = m_e va$, $I = eL/2\pi a^2 m_e$, where m_e is the electron mass. The magnetic moment of the atom is then just $\mu = \pi a^2 I = eL/2m_e$. Now

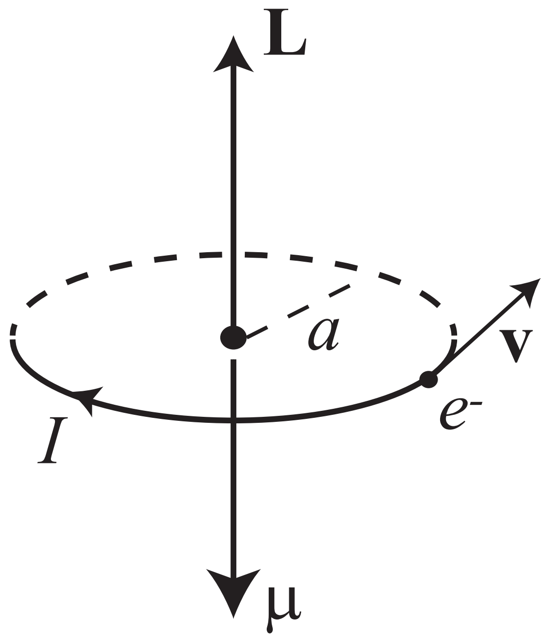

Fig. 7.13 Electron orbiting a nucleus

it turns out that the quantity L cannot take any value, but is quantized in units of $\hbar = 1.054 \times 10^{-34}$J · s, the (reduced) Planck constant.[8] So the minimum possible value for the atomic magnetic moment is

$$\mu_B = \frac{e\hbar}{2m_e} = 9.274 \times 10^{-24} \text{ A} \cdot \text{m}^2, \tag{7.26}$$

A fundamental quantity called the *Bohr magneton*.

Electrons and many other elementary particles also have an *intrinsic* magnetic moment that is approximately equal to $1\mu_B$. This intrinsic moment is related to another property of elementary particles called *spin* that you will

[8]We say reduced because the Planck constant is 6.626×10^{-34} J · s, and $\hbar = h/2\pi$, the Planck constant reduced by a factor of 2π.

learn about in quantum mechanics[9]

When atomic (or intrinsic) magnetic moments are aligned in a material, the material acquires a bulk magnetic moment and we observe macroscopic magnetism. The alignment can occur in different ways, giving rise to different types of magnetic materials.

Paramagnetism In many materials where each atom carries a magnetic moment, the moment of each atom (considered as a vector, $\boldsymbol{\mu}_B$) points randomly, so that the net moment vanishes. However, when a magnetic field $\mathbf{B}_0$ is applied, the moments each feel a torque $\mathbf{\Gamma} = \boldsymbol{\mu} \times \mathbf{B}_0$ and as a result tend to become aligned with the applied field. Remembering that each magnetic moment behaves like a little bar magnet[10], we see that the field from each magnetic moment *adds* to $\mathbf{B}_0$. The total field $\mathbf{B}$ inside the material is then

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_m, \tag{7.27}$$

where $\mathbf{B}_m$ is the additional field due to the alignment of the atomic magnetic moments. It turns out that, at least provided B_0 is not too large, the additional field created by the alignment of the magnetic moments is proportional to the applied field so that $\mathbf{B}_m = \chi_m \mathbf{B}_0$, where χ_m is called the *magnetic susceptibility*, and

$$\mathbf{B} = (1 + \chi_m)\, \mathbf{B}_0 = \mu_r \mathbf{B}_0, \tag{7.28}$$

where μ_r is called the *relative magnetic permeability*. For the materials we are currently discussing, so-called *paramagnetic materials* $\chi_m > 0$ and $\mu_r > 1$. Paramagnetic media are not magnetised, but become so on the application of a magnetic field. Paramagnetic media include Uranium ($\chi_m = 40 \times 10^{-5}$), Platinum ($\chi_m = 26 \times 10^{-5}$) and Aluminium ($\chi_m = 2.2 \times 10^{-5}$). Since temperature randomises the orientation of the atomic magnetic moments, the magnetic susceptibility reduces with increasing temperature ($\chi_m \sim T^{-1}$, known as Curie's law.). The data for χ_m given above is at 20°C.

Diamagnetism For some materials where the atoms do not have a magnetic moment, applying a magnetic field can drive current loops that will then

[9] Note that a proton, for example has an intrinsic magnetic moment given by $\mu_p = e\hbar/2m_p$ which is nearly 2000 times smaller than the Bohr magneton.

[10] We will see this more explicitly later, when considering the field of a current loop using the Biot-Savart law.

acquire magnetic moments. The field associated with these induced moments is always so as to *oppose* the applied field. We will see how this comes about later when we study Lenz's law and Faraday's law. The effect in this case is to *reduce* the applied field from $\mathbf{B}_0$ to $\mu_r\mathbf{B}_0$, where $\mu_r < 1$, and $\chi_m < 0$. Such media are called *diamagnetic*. Examples include Bismuth $\chi_m = -16.6\times10^{-5}$, Mercury $\chi_m = -2.9\times10^{-5}$, Silver $\chi_m = -2.6\times10^{-5}$, Carbon (diamond) $\chi_m = -2.1\times10^{-5}$, Lead $\chi_m = -1.8\times10^{-5}$ and Copper $\chi_m = -1.0\times10^{-5}$. Diamagnetic susceptibilities are very nearly independent of temperature. Having said that, there is an important exception. **Superconductors** are materials that, below their so-called critical temperature, become perfectly conducting. The induced currents when exposed to a magnetic field set up a dipole moment which opposes the field, and the resultant magnetic field inside the superconductor is *zero*. Superconductors are therefore effectively perfect diamagnets and have a susceptibility $\chi_m = -1$, and $\mu_r = 0$.

Ferromagnetism In Ferromagnetic materials the atomic magnetic moments (each of the order of one Bohr magneton, μ_B) become co-aligned over many thousands of atoms forming regions, called *domains*, that each have a high magnetic moment - see Fig. 7.14 Before any external field is app-

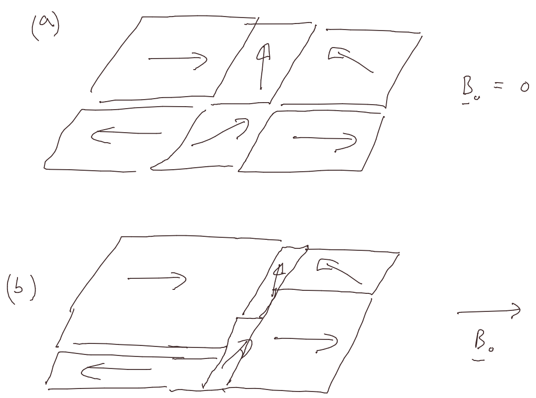

Fig. 7.14 Domains in a Ferromagnetic material: (a) with no external field, (b) with an external field

lied the domains are randomly oriented so that the overall magnetic moment is zero. On applying a field, the domains aligned with the field grow in size, whilst the others diminish (some domain reorientation can also occur). The domain boundaries can thus move until nearly all the material is magnetised. The resultant magnetic susceptibility is much larger than for paramagnetic and diamagnetic media; typically $\chi_m \sim 10^3 - 10^5$. In fact, the value of χ_m for ferromagnetic media depends on the magnitude of $\mathbf{B}_0$, $\chi_m(B_0)$. On removing the external field the material *remains* magnetised. The fact that the magnetisation depends on the history of the applied field is called hysteresis, as illustrated in Fig. 7.15. Examples of ferromagnetic

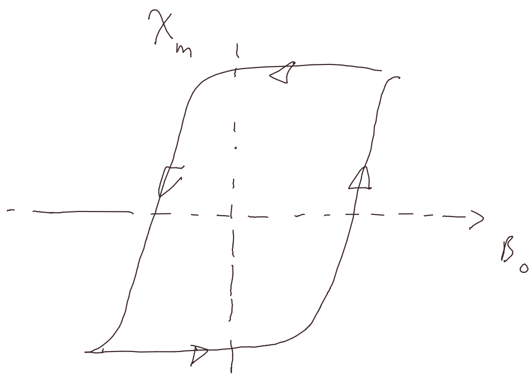

Fig. 7.15 Hysteresis: the magnetisation depends on the history of the applied field. The curve is called a *hysteresis loop*.

materials are Iron and Nickel. A permanent magnet requires a material with a broad hysteresis loop as a significant reverse field is required to demagnetize it. A transformer on the other hand, which is a device for transferring magnetic energy from one circuit to another, uses a ferromagnetic material to enhance the stored magnetic field energy. It is usually the case that the field enhancement should follow changes in the applied magnetic field, and this in turn requires a medium with a thinner hysteresis loop (e.g. iron).

7.3.5 ***How magnets pick up nails***

We are now in a position to answer this deceptively simple question. Being made of iron, nails are ferromagnetic. When the pole of a magnet is brought near, the majority of the domains align with the magnet's magnetic field giving a resultant moment in the same direction as the field as shown in Fig. 7.16(a).

We can simplify by assuming that the resultant magnetic moment of the nail

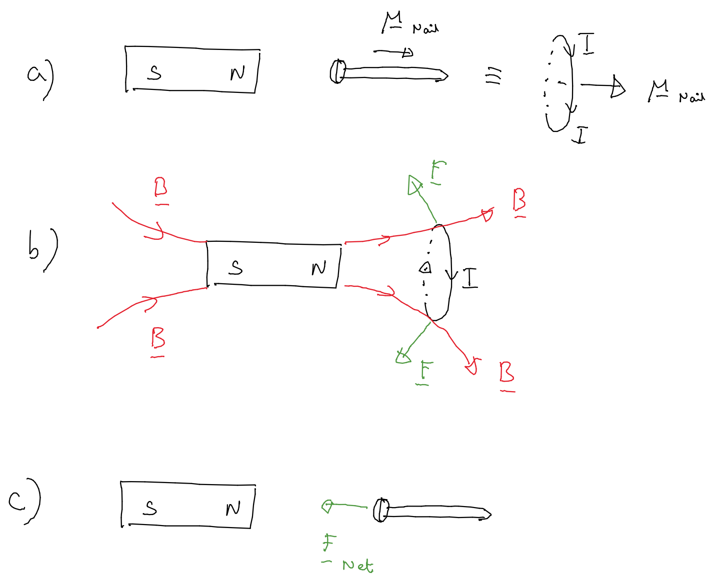

Fig. 7.16 How magnets attract nails.

is the same as that produced by a single loop of current. It is this 'current-loop-equivalent' picture that is the key to understanding the force of attraction that results. Now we know the shape of the magnet's magnetic field from the iron filings experiment, so that the equivalent situation is as illustrated in Fig. 7.16(b). By examining the forces on different parts of the current loop, we see that all the forces have a component acting *towards* the magnet, and a component acting radially away from the loop axis. The latter cancel out when averaged around the whole loop, leaving a net force of attraction towards the magnet (Fig. 7.16(c)). A similar qualitative picture is correct for a magnet being attracted to the 'fridge door. It would be easy from the outset to combine the iron filings experiment with the simple observation that nails are attracted to magnets, and assume that the force of attraction can be identified with the line of the magnet's field lines. As we have seen, that would have been incorrect. The true picture is much more subtle.

7.3.6 *Discussion questions*

(1) If nails are attracted to magnets, why do iron filings not move along the field lines and all end up at the magnet's poles?
(2) If ferromagnetic nails are attracted to a bar magnet, what force results when a superconductor is brought up to a bar magnet? Can you think of an application of this effect? For some amusing possibilities, see Frog levitation and Mouse levitation
(3) How might a loop of wire be used as a compass. Could such a compass distinguish between North and South?

7.3.7 *Numerical Exercises*

(1) The plane of a 5.0 cm × 8.0 cm rectangular loop of wire carrying a current of 0.5 A is parallel to a 0.19 T magnetic field. (a) What torque acts on the loop? (b) What is the magnetic moment of the loop? (c) What is the maximum torque that can be obtained with the same total length of wire carrying the same current in this magnetic field?

7.4 Magnetic Flux, Φ_B

We now know that there is a magnetic field associated with bar magnets, and how that field relates to the motion of charged particles. We will also see quite soon how magnetic fields arise from currents. A couple of words of caution. The magnetic field is not a *force* field, in the sense that the electric field describes the direction of the force on a charged particle. The magnetic force is adduced from the field by the formula $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$. The other point to make is that if we follow magnetic field lines then if they originate from currents then they do not appear to end - rather they form closed *loops* - see Fig. 7.17.

Magnetic fields associated with magnets *do* appear to start at the North pole and end at the South pole. However, we also notice from Fig. 7.17 the great similarity between the field from a bar magnet and that of a solenoid. For the latter the field lines can be traced *through* the solenoid, suggesting that were we able to access the field lines inside the bar magnet they too would form closed loops. The similarity also suggests that the bar magnet actually consists of something equivalent to a solenoidal current.

Now something suggests itself here in terms of comparing electric and magnetic fields. We recall from the electricity part of the course that we can calculate the total electrical flux passing through a closed surface as

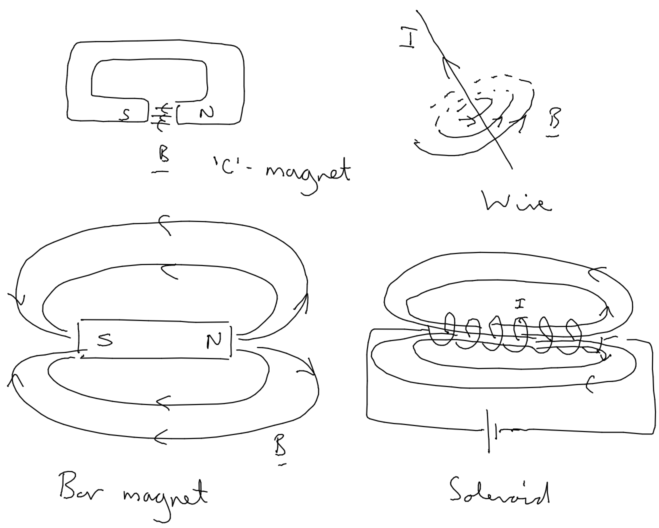

Fig. 7.17 Examples of magnetic fields around magnets and currents

$$\textit{Electrical flux through } S = \oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_T}{\epsilon_0} \tag{7.29}$$

where Q_T is the total charge contained within the volume surrounded by the closed surface, S. We remember that electric fields *do* have a beginning (i.e. they do *not* form loops); they emanate from positive charges (or end on negative charges). Since magnetic field lines do not appear to have a beginning or an end, we propose the analogous equation

$$\textit{Magnetic flux through } S = \oiint_S \mathbf{B} \cdot d\mathbf{S} = 0. \tag{7.30}$$

If correct, then this equation is a mathematical way of expressing the idea that there is no source of magnetic 'charge'. For a bar magnet (or a solenoid), the field is formed by the combination of two opposite magnetic poles, i.e. a dipole. The absence of a 'total charge' term on the rhs of Eq. (7.30) is the mathematical way of expressing the idea that there are no magnetic monopoles. It is then a matter of experiment to decide whether this is in fact the case. Although famous physicists like Dirac found compelling *theoretical* arguments that magnetic monopoles should exist, none have been definitively found in

Nature[11]. We therefore conclude that Eq. (7.30) expresses mathematically the *experimental* observation that there are no magnetic monopoles.

We can use the Divergence Theorem to express Eq. (7.30) differently:

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{B} \, dV \tag{7.31}$$

where S encloses V. But since the r.h.s. of this equation is zero for arbitrary volumes we have that

$$\nabla \cdot \mathbf{B} = 0. \tag{7.32}$$

This is the differential form of Eq. (7.31). It constitutes one of the four famous Maxwell equations that together summarise the whole of electromagnetic theory.

We obtain a non-zero magnetic flux if the surface over which the flux integral is calculated is not closed. We therefore define the Magnetic Flux as

$$\Phi_B = \iint_R \mathbf{B} \cdot d\mathbf{S}. \tag{7.33}$$

The units of magnetic flux are tesla m^2 ($T\ m^2$), or **weber** Wb. The dimensions of magnetic field are therefore alternately expressed as $Wb \cdot m^{-2}$. For this reason **B** is sometimes referred to as the **magnetic flux density**.

7.5 The Magnetic Field Produced by Currents

We have seen how a magnetic field exists around a bar magnet, and how a moving charge is affected by a magnetic field. Moving charges constitute currents, so a wire carrying a current feels a force in a magnetic field. But by Newton's third law, we anticipate that the magnet producing a magnetic field will also feel an equal and opposite force *due to some influence of the current carrying wire on the magnet*. We know that two magnets placed near each other feel a mutual magnetic interaction between them, so it does not seem to be too big a step to suppose that the influence of the current-carrying wire on a bar magnet is of magnetic origin. We can test this by placing iron filings near a current-carrying wire. Here is a YouTube demonstration:

[11] If you are interested, my colleague, Prof. Arttu Rajantie, has written a beautiful (and accessible!) article titled 'Introduction to magnetic monopoles' that is available via the link

Link to Prof. Rajantie's paper

YouTube clip of iron filings placed near a current-carrying wire

So the magnetic field lines *circulate* around the wire - see Fig. 7.18 If the little

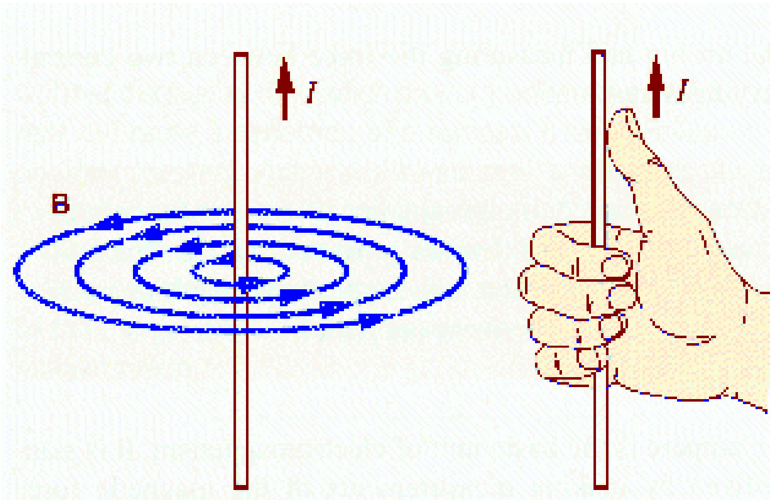

Fig. 7.18 Magnetic field around a current-carrying wire

magnets that make up the iron filings in the YouTube demonstration had North and South labelings, we would be able to track the orientation of the magnetic field as it circulates around the wire. We would find that the field is oriented as shown in Fig. 7.18, i.e. according to the right-hand grip rule.

Now we want to figure out quantitatively the relationship between the current in a wire and the magnetic field it produces. The first thing we can do (conceptually) is to measure the force experienced by a charge moving parallel to the current-carrying wire. Suppose the charge is negative and moves in the opposite direction to the current flow. According to the Lorentz force law, the charge will curve *towards* the wire. Now ultimately experiment determines what the magnetic field is around a wire, and our job as physicists is to find a simple mathematical description. However, rather than just give the answer, I think it's a bit more fun to see how the magnetic field around a wire comes about by just thinking about *electric fields*. The detailed calculation requires Special Relativity, so we will just give a qualitative overview here.

Firstly, we state the Principle of Relativity:

Physics looks the same in all frames in uniform relative motion.

Now just using this principle, and what we already know about electric fields, we will now show that there *must* be a some influence field around a current-carrying wire, and it *must* be directed as shown in Fig. 7.18. Here's how it goes. Consider the situation shown in Fig. 7.19. On the left is the situation

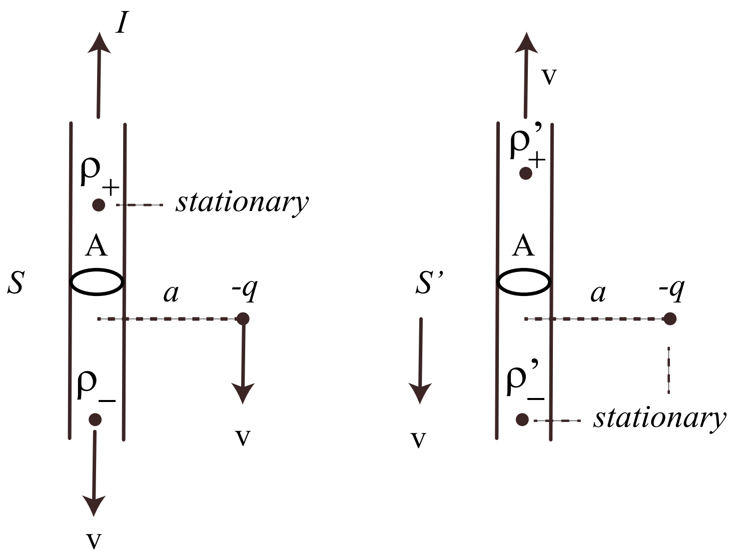

Fig. 7.19 Current-carrying wire and a (negative) charge, viewed from two frames.

similar to Fig. 7.18. All we have done is to add a negatively charged particle moving downwards with speed v. We will call this the perspective from the Lab frame, S. Since the current is upwards, and we assume a metallic wire, the current consists of electrons of charge density ρ_- moving downwards with speed v, and static ions of charge density ρ_+. Since the wire is overall neutral, $\rho_+ + \rho_- = 0$. Now let's consider the same situation from the perspective of an observer who is moving downwards with speed v as shown on the right. We call this frame S'. According to this observer, the electrons are stationary with charge density ρ'_- and the ions move upwards with speed v and have charge density ρ'_+. Why do we make a distinction between $\rho_\pm$ and $\rho'_\pm$? Because they are not the same! When an object moves with speed v, according to Special Relativity, it is contracted by a factor of $\left(1 - v^2/c^2\right)^{1/2}$ in the direction of motion, so this contraction will affect the charge density in S'. If A is the cross-sectional area of the wire[12] then the total positive charge in S over a length ℓ will be $\rho_+ A\ell$ whilst in S' *the same total charge* is given by $\rho'_+ A\ell'$. But the length in S' is contracted with respect to the length in S (recall $\ell' = (1 - v^2/c^2)^{1/2}\ell$). We then have

$$\rho'_+ = \left(1 - v^2/c^2\right)^{-1/2} \rho_+ \, , \tag{7.34}$$

for the positive ions, and

$$\rho_- = \left(1 - v^2/c^2\right)^{-1/2} \rho'_- \, , \tag{7.35}$$

for the electrons since they are at rest in S' and their density is therefore increased in S. Now we know that the wire is neutral in S so that $\rho_- = -\rho_+$. Hence the total charge density in S' is given by

$$\rho'_+ + \rho'_- = \frac{v^2/c^2}{(1 - v^2/c^2)^{1/2}} \rho_+ \, . \tag{7.36}$$

In other words, the observer in S' sees a *net positive charge density* on the wire! Recalling from earlier in the course (Section 3.4) and the result for the electric field distance a from an infinite line of charge (i.e. $E = \rho A/2\pi\epsilon_0 a$), we see that the electric field in S' felt by the test charge is

$$\mathbf{E} = \frac{v^2/c^2}{(1 - v^2/c^2)^{1/2}} \left(\frac{\rho_+ A}{2\pi\epsilon_0 a}\right) \hat{\mathbf{r}} \, , \tag{7.37}$$

directed radially away from the wire. Our test charge is negative, so it will be *attracted to* the wire. We don't need to worry about any magnetic field in S', since in this frame the test charge is stationary, and will not feel any magnetic force.

Now the principle of relativity comes in to play. If the test charge is attracted to the wire in S' *it must also be attracted to the wire in* S! But in S the wire is electrically neutral, and so the test charge, which remember is moving downwards in S, is not *electrically* attracted to the wire in the Lab frame. The only way it can be deviated towards the wire is if it feels a Lorentz force due to a magnetic field generated by the wire. Furthermore, since the Lorentz force is given by $q\mathbf{v} \times \mathbf{B}$, we can deduce the direction of $\mathbf{B}$ to be as shown in Fig. 7.18.

[12] n.b. the area A is the same in S and S'. It is only lengths parallel to the direction of relative motion that change.

7.5.1 *Ampère's Law*

Now we *could* base our study of magnetic fields from currents on understanding how they arise from 'moving' electric fields, but that would be a bad idea for two reasons. Firstly, we can only get the right answers by carefully applying Special Relativity, and that course comes later. Secondly, and this is perhaps a better answer, we might be faced with a situation where several magnetic fields arise from several currents of different magnitudes and directions. In that case, lots of different moving frames would need to be introduced, and the whole thing would rapidly become complicated. We might as well treat the magnetic field as a separate entity and find out the rules that associate a magnetic field with a current. There are actually two (equivalent) possibilities, one that relates a finite field to a finite current, and one that relates an infinitesimal field to an infinitesimal current.

The first possibility is encapsulated by Ampère's Law which states that in the absence of time-varying electric fields

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{enclosed}, \tag{7.38}$$

where $\mu_0 = 4\pi \times 10^{-7}$ TmA^{-1} is a constant of nature called the **permeability of free space**. In Eq. (7.38) $I_{enclosed}$ means the current that runs through the closed loop C - see Fig. 7.20. The first thing to note is that the value of the

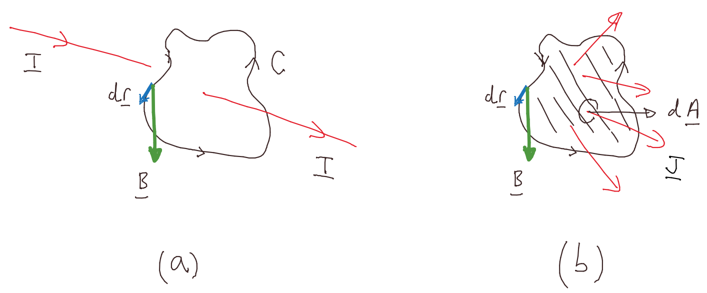

Fig. 7.20 Ampè's Law (a) single current, (b) current density.

line integral $\oint_C \mathbf{B} \cdot d\mathbf{r}$ is independent of the shape of C. So long as C encloses the current $I_{enclosed}$, the value of the line integral is just $\mu_0 I_{enclosed}$. Next, what meaning is to be given to the sign of $I_{enclosed}$? We will use the convention that says a positive current is to be associated with the right-hand rule for the circulation C, i.e. in Fig. 7.20(a) $I_{enclosed} > 0$. Finally, the 'current enclosed' means the total current, so that if more than one current runs through C then we must add them all together to get the total $I_{enclosed}$. This shows, incidentally, that the **B**-field, like the **E**-field follows the principle of superposition. In fact, as indicated in the Fig. 7.20(b), there could be a continuous current *density* running through C in which case Eq. (7.38) becomes

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \iint_A \mathbf{J} \cdot d\mathbf{A}, \tag{7.39}$$

where the area A encloses C. Actually, there are many different areas that can enclose C but it turns out, remarkably, that it doesn't matter which one we choose. The only constraint is that for a given area, the direction associated with each area element $d\mathbf{A}$ must be chosen with respect to the right-hand rule for the direction of circulation C. Also, does the sense of circulation around C matter? No, because if we reverse the sense of circulation, then, with respect to the new sense of circulation, we reverse the current and everything cancels.

Ampère's law is often a good way to calculate the magnetic field in cases of high symmetry, where we can guess the direction of the field, and deduce that its magnitude doesn't change around a loop chosen to respect the symmetry of the problem. Let's work through a few examples.

7.5.2 *Some simple calculations of magnetic field*

7.5.2.1 *Long Wire*

This is the simplest example - see Fig. 7.21. The magnetic field at radius r

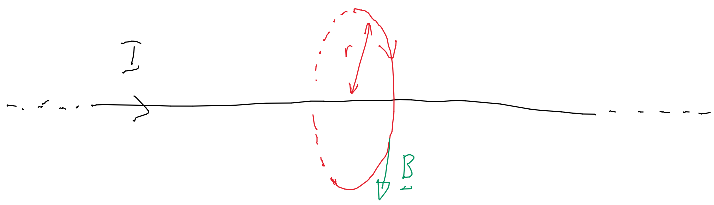

Fig. 7.21 Calculation of the magnetic field around a long wire using Ampé's Law.

from the wire is directed tangentially and is of constant magnitude. Therefore we can immediately evaluate Eq. (7.38) as

$$2\pi r B = \mu_0 I\,, \quad \Rightarrow B = \frac{\mu_0 I}{2\pi r}\,. \tag{7.40}$$

The direction of **B** is determined by the right-hand rule.

7.5.2.2 *Solenoid*

Now let us calculate the magnetic field inside a long solenoid - see Fig. 7.22. We choose an Ampèrian circuit ABDC as shown where $AB = \ell$ is parallel to

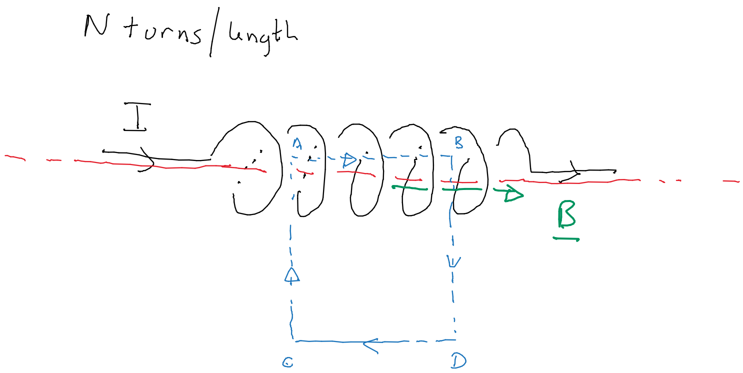

Fig. 7.22 Calculation of the magnetic field around a long wire using Amp'e's Law.

the axis and there are N turns per unit length. By long we mean that in the region of interest (the line AB in the diagram), the field doesn't vary very much. To calculate the field using Ampère's Law we can note that

- by symmetry $\int_C^A \mathbf{B}\cdot d\mathbf{r} = -\int_B^D \mathbf{B}\cdot d\mathbf{r}$,
- the lines CA and BD can be made so long that $\int_D^C \mathbf{B}\cdot d\mathbf{r} = 0$,
- the current enclosed by the loop is $NI\ell$.

Ampère's law therefore becomes

$$\oint \mathbf{B}\cdot d\mathbf{r} = \int_A^B \mathbf{B}\cdot d\mathbf{r} = B\ell = \mu_0 NI\ell, \tag{7.41}$$

and hence $B = \mu_0 NI$ directed along the axis according to the right-hand rule. Note that the result is independent of how AB is positioned within the solenoid; it does not necessarily have to lie on the axis, just parallel to it. Hence the field inside the solenoid is constant

7.5.3 *Differential form of Ampère's law*

Using Stokes' theorem

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 I_{\text{enclosed}} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S}. \qquad (7.42)$$

Hence (for static fields)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \qquad (7.43)$$

This is *nearly* one of Maxwell's equations. Maxwell, via a brilliant theoretical argument, deduced that there should actually be another term present on the right-hand side ($= \epsilon_0 \mu_0 \partial \mathbf{E}/\partial t$), that couples a *time-varying* electric field to the magnetic field. We will consider this extra term a bit later.

7.5.4 *The Magnetic Field of a Current Element - the Biot-Savart Law*

Ampère's law above (Eq. (7.38)) can often be used to calculate the magnetic field in situations of high symmetry (cf. Gauss' law of electrostatics). For more general situations, such as for the arbitrarily shaped current loop in Fig. 7.23 Ampère's law is unsuitable. What we need is a way of breaking up the problem

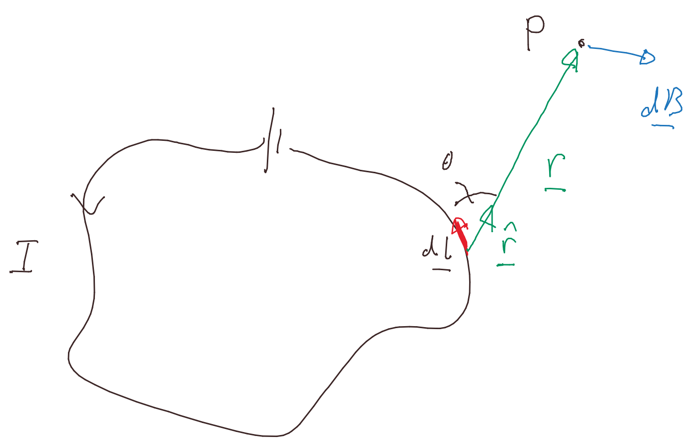

Fig. 7.23 Magnetic field of an arbitrarily shaped current loop.

into little pieces and adding them together using the superposition principle. In the figure we identify a *current element* of (vector) length $d\mathbf{l}$. Say we want to figure out the magnetic field at point P, where $\mathbf{r}$ is the displacement from the base of $d\mathbf{l}$ to the point P. We imagine that the vector $d\mathbf{B}$ at P is the infinitesimal magnetic field just due to the current running through $d\mathbf{l}$. If we know how to relate $d\mathbf{B}$ to $d\mathbf{l}$ and $\mathbf{r}$ then we could find the total field at P by adding up (i.e. integrating) all the $d\mathbf{B}$'s due to all the $d\mathbf{l}$'s around the circuit. The required relationship is known as the Biot-Savart law. Whilst we can't *per se* derive the Biot-Savart law (ultimately it's experiments that decide), we can obtain a very plausible *idea* of what the law must look like by making some reasonable assumptions and ensuring compatibility with Ampère's law.

From the principle of superposition, if we double the length of $d\mathbf{l}$ then we double the amount of current in $Id\mathbf{l}$, so we expect the field strength $d\mathbf{B}$ to then also double. So we anticipate $|d\mathbf{B}| \propto I\,|d\mathbf{l}|$. It also seems reasonable to expect that the contribution to the magnetic field at P depends as some power law on the distance $r = |\mathbf{r}|$; i.e. $|d\mathbf{B}| \propto r^n$ for some integer n. Finally, any dependence on θ, the angle between $d\mathbf{l}$ and $\mathbf{r}$, can be absorbed into some to-be-determined function $f(\theta)$. So our proposed law for $dB = |d\mathbf{B}|$ looks like

$$dB = kIr^n f(\theta) dl, \tag{7.44}$$

where $dl = |d\mathbf{l}|$ and k is a constant of proportionality. In order to try and figure out the value of n, and what $f(\theta)$ might be, we need to apply the above expression to calculate the magnitude of $\mathbf{B}$ for a situation where we know the answer. We know that at a distance a from an infinitely long wire $B = \mu_0 I/2\pi a$ (see Eq. (7.43)). Using the geometry of Fig. 7.24 we see that our proposed

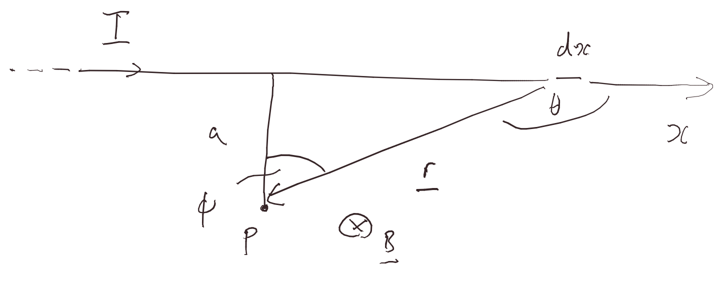

Fig. 7.24 Magnetic field near an infinite wire - towards the Biot-Savart law.

law must satisfy

$$B = \int_{-\infty}^{\infty} kIr^n F(\phi) dx = \frac{\mu_0 I}{2\pi a}, \tag{7.45}$$

where we have set $f(\theta) = F(\phi)$ (since ϕ is a slightly more convenient variable to work with for the moment), and $dx = dl$. Converting the integral into one over the variable ϕ we note that

$$x = a \tan\phi, \qquad dx = \frac{a}{\cos^2\phi} d\phi, \qquad r = \frac{a}{\cos\phi} \tag{7.46}$$

so that Eq. (7.45) becomes

$$kIa^n \int_{-\pi/2}^{\pi/2} \frac{F(\phi)}{\cos^n\phi} \frac{a}{\cos^2\phi} d\phi = \frac{\mu_0 I}{2\pi a}. \tag{7.47}$$

In order to get the powers of a to balance, n must be equal to -2. Comparing with Eq. (7.44) we see that like all venerable field theories in physics, the magnetic field is an inverse square law. The magnetic field falls off as the inverse square of the distance (r) from the source. Equation (7.47) then reduces to

$$k \int_{-\pi/2}^{\pi/2} F(\phi) d\phi = \frac{\mu_0}{2\pi}. \tag{7.48}$$

If we set $k = \mu_0/2\pi$, then the above equation implies that

$$\int_{-\pi/2}^{\pi/2} F(\phi) d\phi = 1. \tag{7.49}$$

Of course there are many possible functions that can satisfy this. But it seems reasonable that since ϕ (or ultimately θ) represents an angular dependence between $d\mathbf{l}$ and $\mathbf{r}$, it is likely to be some trigonometric function, like $\sin\phi$, $\cos\phi$, or $\tan\phi$. If we restrict to these simplest possibilities, we can reject $\sin\phi$ (because that gives zero when integrated between $\pm\pi/2$), and $\tan\phi$ (because that gives infinity), and we are left with $F(\phi) = \frac{1}{2}\cos\phi$:

$$\frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos\phi d\phi = \frac{1}{2} [\sin\phi]_{-\pi/2}^{\pi/2} = 1. \tag{7.50}$$

From the geometry of Fig. 7.24 $\phi = \theta - \pi/2$, so $\cos\phi = \sin\theta$, and $f(\theta) = \frac{1}{2}\sin\theta$. Equation (7.44) becomes, when we put everything together

$$dB = Ir^n f(\theta)dl = \frac{\mu_0}{2\pi} I \frac{1}{r^2} \frac{\sin\theta}{2} dl = \frac{\mu_0 I}{4\pi r^2} \sin\theta dl. \quad (7.51)$$

In fact, if we bear in mind our example of the infinite wire in Fig. 7.24, where $d\mathbf{l} = \mathbf{i}dx$ we see that we will get the right answer for the *direction* of $\mathbf{B}$ if we make each contribution $d\mathbf{B}$ point in the direction $d\mathbf{l} \times \hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector directed along $\mathbf{r}$. So we can transform Eq. (7.51) into a vector equation by setting

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \mathbf{r}}{r^3}. \quad (7.52)$$

This provides a way of relating an infinitesimal magnetic field to an infinitesimal current, that is consistent with Ampère's law. In fact it turns out (experimentally) to be *the* rule that works generally, and is known at the Biot-Savart Law. To calculate the total magnetic field at the point P we must evaluate the integral

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \quad (7.53)$$

If you find the positioning of the integration variable ($d\mathbf{l}$) a bit strange in the above expression, then it will likely become clearer when working through the following example.

Example: calculate the magnetic field on the axis of a circular current loop of radius a. From the geometry of Fig. 7.25

$$d\mathbf{l} = -a\sin\phi d\phi\mathbf{i} + a\cos\phi d\phi\mathbf{j}, \quad \mathbf{r} = -a\cos\phi\mathbf{i} - a\sin\phi\mathbf{j} + z\mathbf{k}, \quad r = \left(a^2 + z^2\right)^{1/2}, \quad (7.54)$$

so that

$$d\mathbf{l}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -ad\phi\sin\phi & ad\phi\cos\phi & 0 \\ -a\cos\phi & -a\sin\phi & z \end{vmatrix} d\phi = azd\phi\cos\phi\mathbf{i} + azd\phi\sin\phi\mathbf{j} + a^2 d\phi\mathbf{k}. \quad (7.55)$$

The Biot-Savart integral then becomes

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_C \frac{az\cos\phi\mathbf{i} + az\sin\phi\mathbf{j} + a^2\mathbf{k}}{(a^2 + z^2)^{3/2}} d\phi. \quad (7.56)$$

This is three separate integrals. The first two vanish, whilst the third just yields a factor of 2π, so that

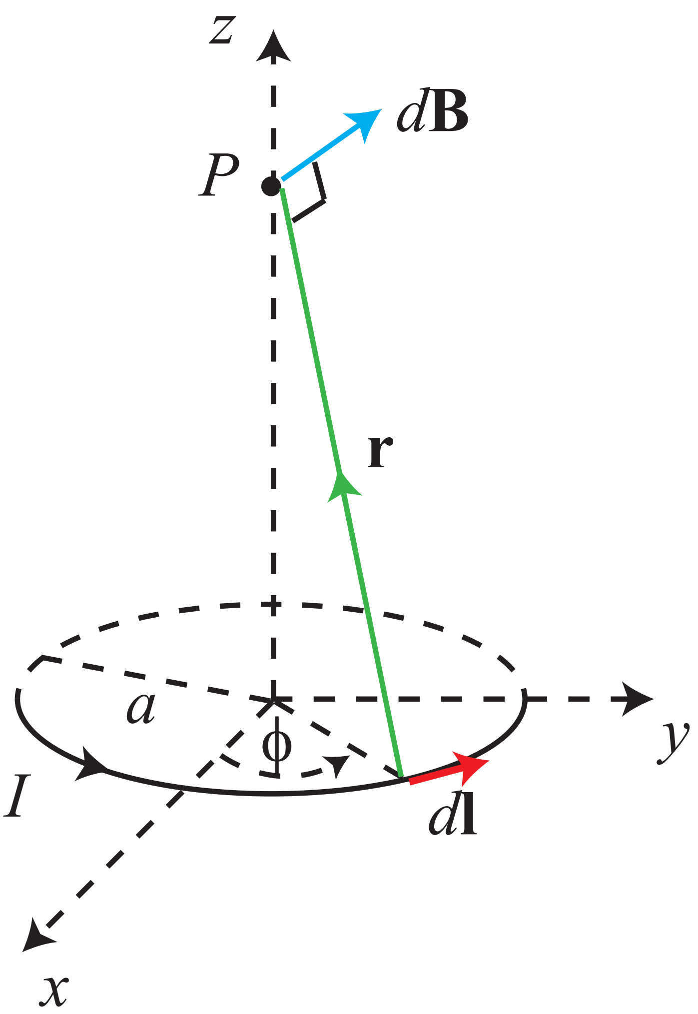

Fig. 7.25 Calculation of the magnetic field on the axis of a current loop using the Biot-Savart law.

$$\mathbf{B} = \frac{\mu_0 I a^2}{2\,(a^2 + z^2)^{3/2}}\mathbf{k}. \tag{7.57}$$

In terms of the magnetic moment $\boldsymbol{\mu} = \pi a^2 I\mathbf{k}$ this may be written as

$$\mathbf{B} = \frac{\mu_0}{2\pi\left(a^2 + z^2\right)^{3/2}}\boldsymbol{\mu}. \tag{7.58}$$

At $z = 0$ and $z \gg a$ we have respectively

$$\mathbf{B}(z = 0) = \frac{\mu_0}{2\pi a^3}\boldsymbol{\mu}, \quad \text{and} \quad \mathbf{B}(z \gg a) = \frac{\mu_0}{2\pi z^3}\boldsymbol{\mu}. \tag{7.59}$$

The latter expression should be compared with the electric field along the axis of an *electric* dipole $\mathbf{E} = \mathbf{p}/4\pi\epsilon_0 z^3$ where $\mathbf{p} = q\mathbf{l}$ is the *electric* dipole moment.

7.5.5 *Forces between wires*

We now know three things:

(1) a current produces a magnetic field (cf. Eq. (7.43) and Fig. 7.21).
(2) a current consists of moving charges (cf. Eq. (6.2) and Fig. 6.1).
(3) moving charges feel a force in a magnetic field (cf. Eq. (7.1) and Fig. 7.2).

Two parallel wires must feel a force between them because whilst one of the wires produces a magnetic field, the current in the other wire consists of moving charges. Of course the forces between the two wires are equal and opposite by Newton's third law. Let us calculate this mutual force of interaction using what we now know about magnetic fields and forces. It will be easiest to consider two *infinite* wires distance a apart, as shown in Fig. 7.26. At point

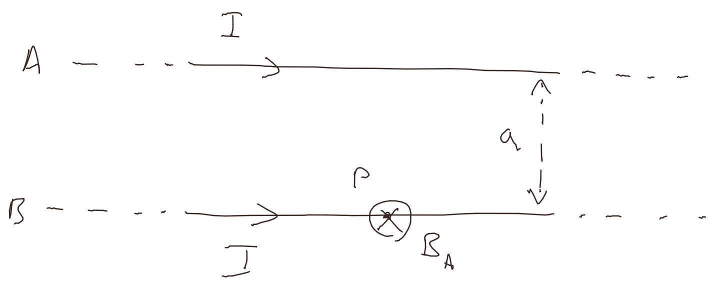

Fig. 7.26 Parallel wires carrying co-directional equal currents

P the magnetic field due to the current in wire A is $B_A = \mu_0 I/2\pi a$, directed as shown in the figure. Since both wires are infinite, there is nothing special about point P so B_A is the magnitude of the magnetic field at all points along wire B. The force on wire B is therefore $F = B_A IL$, where L is the length of wire B. Unsurprisingly, since the wires are of infinite length, the force F is also infinite. It makes more sense to look at the force per unit length

$$F_\ell = \frac{F}{L} = B_A I = \frac{\mu_0 I^2}{2\pi a} \,. \tag{7.60}$$

We can see that the force is attractive either from the Lorentz force law, or from $\mathbf{F} = I\hat{\mathbf{n}} \times \mathbf{B}\ell$. Naturally, this is exactly the force per unit length experienced by wire A due to the magnetic field of wire B. We can see the effect of this force from the following link

Force between parallel wires

7.5.5.1 *Definition of the ampere and coulomb*

The force between current carrying wires is used in the SI system of units to *define* the unit of current, the *ampere*:

> 1 ampere is the constant current that will produce an attractive force of 2×10^{-7} newton per metre of length between two straight, parallel conductors of infinite length and negligible circular cross section placed one metre apart in a vacuum.

One might remark on the abstractness of this definition given that any real wires are of finite length. However, as with many areas of physics the unattainable 'ideal' is useful because it can often be approached with arbitrary precision. The laws of physics and experiments can then combine to see if, in any *real* situation, a contradiction arises. If not, then we carry on with our abstract definition, which may be convenient for other reasons. But why is this abstract definition chosen over an apparently more logical thread that starts with defining the coulomb with respect to the force between charges, and then the ampere as a charge of 1 coulomb passing a given point per second? In fact, the SI units are in the process of revision and it is likely that the ampere will be re-defined in the future based on the charge on the electron. Then a definition of the ampere based on the flow of charge could be realised by means of electron pumps which transport electrons one at a time in a controlled way. Historically, the practical problem with using coulomb's law to define charge is the difficulty of controlling macroscopic quantities of static charge. In practice charge would need to be first put onto some insulators so a force could be measured, and then discharged through conductors *without any of it leaking away in the meantime*. On the other hand, if we are just dealing with the flow

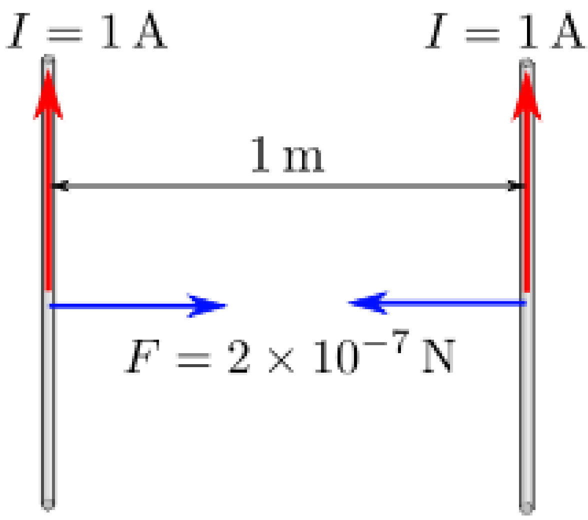

Fig. 7.27 Parallel wires carrying co-directional equal currents: definition of the ampere

of current, we can exploit the fact that wires can be insulated from each other, and to a very high level of accuracy we can control where the current goes, and therefore the magnetic field produced by that current.[13]

With the ampere defined, we can now give the SI definition of the coulomb as:

> 1 coulomb is the quantity of electricity carried in 1 second by a current of 1 ampere

[13] I am very grateful to Dr Stephen Giblin of the Quantum Detection Group at the National Physical Laboratory for providing this information.

Chapter 8

Time Varying Magnetic Fields

8.1 Inducing a Current by Changing a Magnetic Flux

Earlier in the course when we studied static electric fields we saw that the work done by an electric field in taking a charge around a closed loop was zero, i.e.

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = 0. \tag{8.1}$$

Moreover, using Stokes' theorem, we showed that an equivalent statement was

$$\nabla \times \mathbf{E} = 0. \tag{8.2}$$

Now we would like to see how the above rule is modified in the presence of *dynamic* fields and *moving* charges. We will find that the curl of **E** does not vanish but is related to how any magnetic flux present changes with time. Once we have the full equation we will be able to see how a time-varying flux can be used to *generate* an electric field that was not present initially. The induced electric field can then be used to drive charges around circuits, and is therefore equivalent to an induced potential difference, or emf. In fact, most electric currents we encounter in our daily lives are not driven by batteries, but rather by emf's induced by a time varying magnetic flux. The laws of electromagnetic induction were found mostly experimentally through a series of ground-breaking experiments by Michael Faraday. The Maxwell equation we are about to derive is therefore called Faraday's law.

Consider the set-up in Fig. 8.1 which shows a uniform magnetic field directed towards us (i.e. out of the paper). A conducting wire is laid across a pair of parallel wires that are connected at one end via an ameter that records any current flowing. The wire is pushed in the x-direction and moves with constant

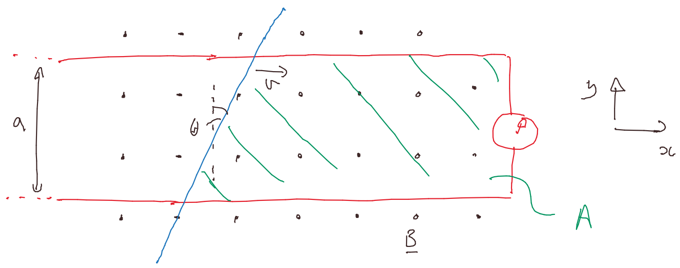

Fig. 8.1 A wire moving in a magnetic field connected to a curcuit.

speed v. At all times contact is maintained with the parallel rails, so that any current that is generated is free to flow around the circuit defined by the rails and the wire.

Q. What do we observe?

A. We observe a constant current that flows in the *anti-clockwise* direction.

Now let's see if we can understand how this comes about. The charges in the wire (presumed positive) move with velocity $\mathbf{v}$ in the x-direction. They therefore feel a Lorentz force of magnitude qvB acting downwards (remember the Lorentz force is given by $q\mathbf{v} \times \mathbf{B}$). Now it is the component of this force $qvB\cos\theta$ that drives current downwards along the wire. Hence the current flows anti-clockwise around the loop.

But we can now go a bit further. If we know the force that is driving the charges, we can calculate the work W done by this force in transporting a charge along the length ($= a/\cos\theta$) of the moving wire:

$$W = F\frac{a}{\cos\theta} = qvB\cos\theta \times \frac{a}{\cos\theta} = qBa\frac{dx}{dt} = qB\frac{dA}{dt}, \tag{8.3}$$

where A is the shaded area shown in the diagram. We are thus led to the important result that the work done per unit charge (i.e. the induced emf, $\mathcal{E}$!), is just

$$\mathcal{E} = \frac{W}{q} = B\frac{dA}{dt} = \frac{d\Phi_B}{dt}, \tag{8.4}$$

where $\Phi_B = BA$ is the the magnetic flux passing through the enclosed area. If the circuit has an overall resistance of R then the current is just $I = R^{-1}d\Phi/dt$, and the power dissipated in the circuit is

$$P = I\mathcal{E} = \frac{1}{R}\left(\frac{d\Phi_B}{dt}\right)^2. \tag{8.5}$$

Now you may have noticed something curious here. By setting the wire in motion with constant velocity, we have induced an emf in the circuit which drives a current. Power is dissipated through the circuit according to Eq. (8.5). But according to Newton's first law, in the absence of any resistance, the wire will just keep moving with constant velocity. Haven't we just found a way of generating power out of nothing?! Before reading on, have a think about what might be happening here.

All is revealed in Fig. 8.2.

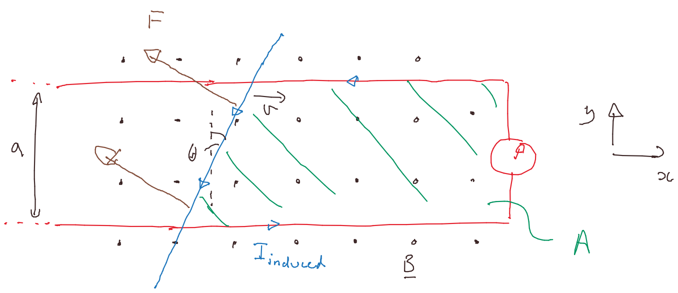

Fig. 8.2 Current induced in a wire moving in a magnetic field, showing the Lorentz force experienced by the induced current

The figure now shows that with the induced current I running down the moving wire, there will be a *new* Lorentz force acting in the direction shown. Its magnitude is given by

$$F = IB\frac{a}{\cos\theta}, \tag{8.6}$$

(recall that the Lorentz force on a wire of length ℓ is given by $I\hat{\mathbf{n}} \times \mathbf{B}\ell$; in this case $\ell = a/\cos\theta$). So the reality is that we cannot set the wire into uniform motion along the x-direction, without pushing against the Lorentz force arising from the interaction of the induced current with the magnetic field. We can calculate the rate at which we must work when pushing against this force:

$$P = -\mathbf{F}\cdot\mathbf{v} = Fv\cos\theta = IB\frac{a}{\cos\theta} \times v\cos\theta = IBav = IBa\frac{dx}{dt} = I\frac{d\Phi_B}{dt} = \frac{1}{R}\left(\frac{d\Phi_B}{dt}\right)^2. \tag{8.7}$$

This is *exactly* equal to the power dissipated in the circuit (see Eq. (8.5))! When we push on the wire, we feel an electromagnetic resistance. The work we do against that resistance is converted into Ohmic dissipation in the circuit.

8.2 Lenz's Law and Faraday's Law of Induction

What we have deduced above is easily generalized. Firstly it is not hard to show that the shape of the circuit is irrelevant, all that matters is the amount of magnetic flux that flows through (or 'links') the circuit, and how this flux varies with time. Secondly, experiments show that the flux can either be varied by changing the area in a uniform magnetic field, or keeping the area constant and varying the magnetic field (or some combination of both). Again all that matters is the flux, which according to Eq. (7.33) is given by

$$\Phi_B = \iint_R \mathbf{B} \cdot d\mathbf{S}. \tag{8.8}$$

Then, the emf induced around any circuit linked by the flux Φ_B is given by

$$\mathcal{E} = \frac{\partial \Phi_B}{\partial t}. \tag{8.9}$$

We now use partial derivatives because it might be the case that **B** depends on space, and we want to make sure we only include time variations of **B** at fixed points in space. In the case studied above we could calculate the magnitude of the induced emf from Eq. (8.9) and deduce the direction of the induced current by thinking about the Lorentz forces on the charges comprising the enclosing circuit. It is better, however, if we do the job more generally by writing instead of Eq. (8.9)

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t}, \tag{8.10}$$

where the *meaning* of the minus sign at this point is that it is a shorthand for saying that

> The induced electromotive force (emf) always gives rise to a current whose magnetic field opposes the original change in magnetic flux.

This statement is known as **Lenz's Law**, and Eq. (8.10) is known as **Faraday's Law of Induction**.

If you think about it, it doesn't matter which direction we assign to the vector $d\mathbf{S}$ in Eq. (8.8). Consider Fig. 8.3. which shows a conducting loop in the

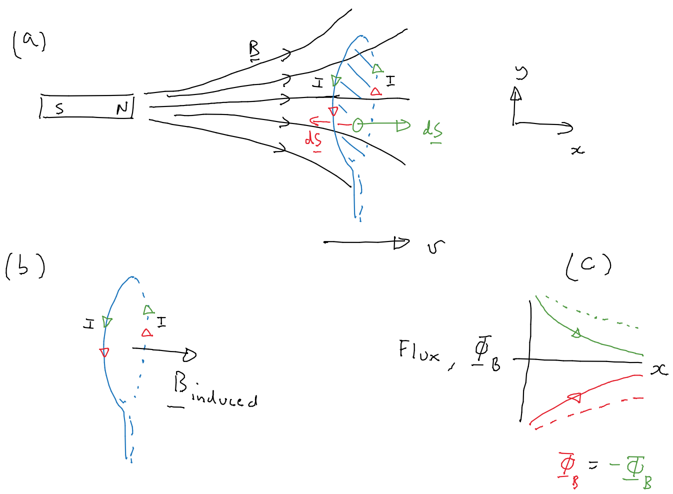

Fig. 8.3 Illustrating that the direction of the induced current in Lenz's law is insensitive to the direction defining $d\mathbf{S}$ in $\iint_R \mathbf{B}\cdot d\mathbf{S}$

magnetic field of a bar magnet. In which direction does the current flow as we move the loop to the right as shown? Now the sign of the flux depends on the direction we assign to the area elements $d\mathbf{S}$ comprising the area within the loop. If $d\mathbf{S}$ points in the same direction as $\mathbf{B}$ then the flux Φ_B is positive and diminishes towards zero as the loop is moved to the right. According to Lenz's law, the induced current must flow so as to generate a magnetic field that *opposes* this decrease in flux. To make the flux increase (cf. the upper dotted curve in Fig. 8.3(c)) the induced magnetic field must be in the same direction as the original **B**-field, and this can only come about if the current flows in the direction shown in Fig. 8.3(b). Now let's run the argument with the direction of $d\mathbf{S}$ defined to *oppose* the original **B**-field. Now Φ_B starts off being negative, but tends to zero as the loop moves to the right (see lower curve in Fig. 8.3(c)). Attempting to sustain the flux at this negative value (cf. lower dotted curve) requires the induced **B** to *again* point to the right; and again the direction of induced current flow is as shown in Fig. 8.3(b). The direction of induced current flow does not depend on our choice the direction of $d\mathbf{S}$, which is as it should be.

8.3 Examples of Electromagnetic Induction

8.3.1 *Dynamo*

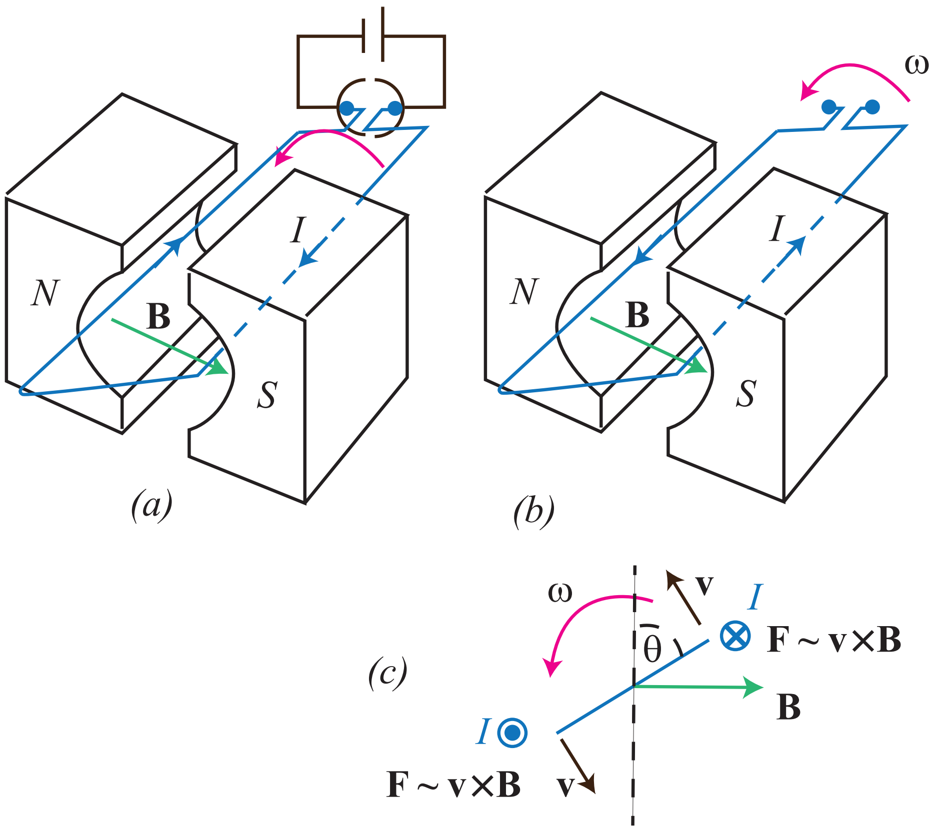

Fig. 8.4 a) DC Motor, b) Dynamo, c) Schematic side view of dynamo

Our first example of electromagnetic induction is the *dynamo* which is the reverse of an electric motor - see Fig. 8.4. Whereas previously (Sec. 7.3.3) we showed how passing a current through a loop placed in a uniform magnetic field caused the loop to rotate (Fig. 8.4(a)), here we will analyse the effect of manually rotating the loop (Fig. 8.4(b)) at a constant angular speed ω when it initially does not carry a current. If the area of the loop is A, then the magnitude of the magnetic flux linking the loop is $\Phi_B = A\cos\theta B$, where θ is the angle between the current loop and the vertical as shown in Fig. 8.4(c). Bearing in mind that as shown θ is decreasing, so that the angular speed at which the loop is cranked round is $\omega = -d\theta/dt$, the rate of change of flux is

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}A\cos\theta B = -AB\sin\theta\frac{d\theta}{dt} = \omega AB\sin\theta\,. \tag{8.11}$$

Faraday's law of induction says that this rate of change of flux equates to an induced emf in the loop. The induced emf drives a current round the loop, and we can deduce the direction of the current from Lenz's law. In the situation illustrated the flux is *increasing* so the induced current must be so as to diminish the flux. The current must therefore be flowing as shown in Fig. 8.4(b) since this will produce a magnetic field opposite to the field from the magnet. We can also deduce the direction of current from the Lorentz force law. As the coil rotates, the velocity of the charges in the upper and lower wire sections are as indicated in the side view of Fig. 8.4(c). Looking at $\mathbf{v} \times \mathbf{B}$ we see that the direction of the induced current in these two wire section checks out with Lenz's law.

The variation of the flux and induced emf is sketched in Fig. 8.5. Note

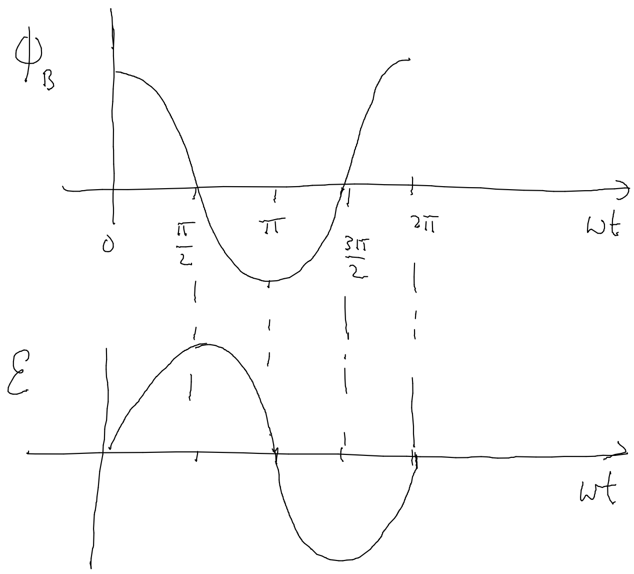

Fig. 8.5 Flux, $\Phi(t)$ and induced emf $\mathcal{E}(t)$.

that when the flux Φ_B is a maximum or a minimum, the induced emf $\mathcal{E}$ is zero.

Since the current in the wire is proportional to the induced emf $I = \mathcal{E}/R$, we can also deduce that as the loop is rotated the current *alternates*. Just as for the motor, the alternation can be rectified with a commutator contact, and the current will then only flow in one direction.

8.3.2 *Induced Field Around a Solenoid*

Our next example is to examine what happens outside a solenoid when the solenoidal current is varied - see Fig. 8.6. We imagine a loop of wire surrounding

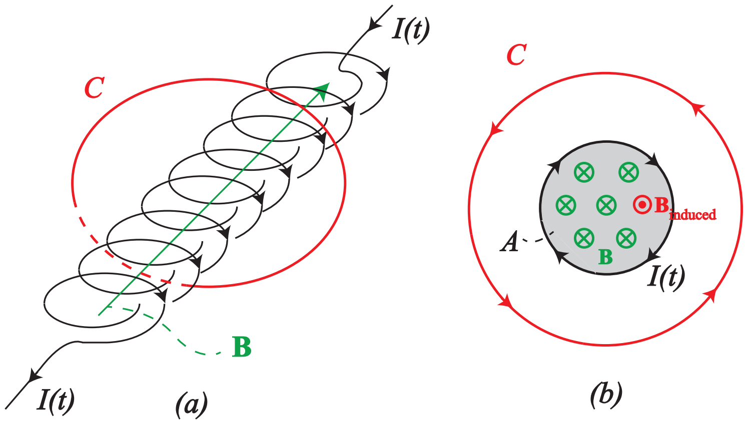

Fig. 8.6 Induced field around a loop surrounding a solenoid

the solenoid as shown. The current through the solenoid creates a magnetic field in the direction shown. We know from earlier work that the field is approximately constant inside the solenoid and is directed along the axis (I chose the direction of the windings deliberately to illustrate that here, the magnetic field inside the solenoid is *oppositely* directed to the current. Always use the right-hand-rule!). Now imagine that the current is varied so that the flux through the solenoid also varies, $\Phi_B(t)$. Let us say that the current is increasing, so that $d\Phi_B/dt > 0$. This flux links the curve C on which the current loop lies, so it must be the case that there is an emf induced around the loop according to Faraday's law. It's not hard to see that this emf must drive a current round the loop as shown in Fig. 8.6(b) since, according to Lenz's law the current must flow so as to induce a magnetic field that opposes the increasing flux. But this example differs from the previous one in a very important respect. For a long solenoid, the field outside is approximately *zero*, so here *we cannot use the Lorentz force argument to deduce the direction of induced current flow*. It must be the case that the induced emf around the loop is the result of some *electric field* that is induced by the time-varying magnetic flux. We then have for the induced emf

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial \Phi_B}{\partial t} \, . \tag{8.12}$$

8.4 Faraday's Law - Differential Form

Using Stokes' theorem, we can obtain a link between the electric field associated with the induced emf, and the rate of change of the magnetic field.

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S}. \tag{8.13}$$

Using Stokes' theorem:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S}. \tag{8.14}$$

Hence

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{8.15}$$

This is the differential form of Faraday's Law. It forms another of Maxwell's equations and is another milestone in our course.

Now whereas previously the sign on the rhs was used as a reminder that the induced emf drove a current so as to oppose changes in flux (cf. Eq. (8.10)), now the sign assumes a greater significance. According to Eq. (8.15) a time-varying magnetic field produces an electric field. The minus sign is now *necessary* to get the correct direction for the induced electric field, **E**.

8.5 Self Inductance

Consider the simple circuit in Fig. 8.7. When the switch is closed a current flows and a magnetic field is generated as shown. Since there was no magnetic field present initially, we know that it must be a time-varying field, and in turn a time-varying magnetic flux Φ_B will link the circuit. According to Lenz's law,

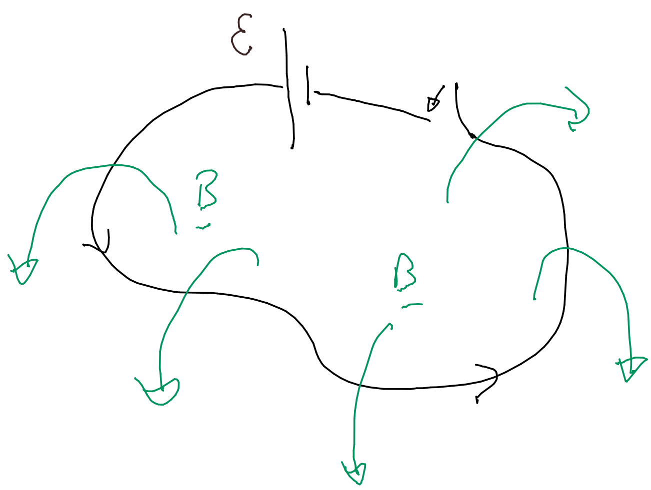

Fig. 8.7 Self-inductance in a simple circuit.

an emf $\mathcal{E}_{ind}$ will be induced in the circuit that will try to oppose the increasing flux - we call it the *back emf.* Now the flux is proportional to the magnetic field, which in turn is proportional to the current flowing through the circuit:

$$\Phi_B(t) \propto |\mathbf{B}| \propto I(t). \tag{8.16}$$

So we can write for the induced emf:

$$\mathcal{E}_{ind} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}, \tag{8.17}$$

where L is a constant of proportionality related to the geometry of the circuit and is called the *self-inductance*. If the total resistance of the circuit is R (the internal resistance of the battery, plus the resistance of the wire) then we have

$$\mathcal{E}_{battery} + \mathcal{E}_{ind} = IR, \quad \text{or} \quad \mathcal{E}_{battery} = IR + L\frac{dI}{dt}, \tag{8.18}$$

where $\mathcal{E}_{battery}$ is the emf supplied by the battery. This is a simple differential equation that you can (and should!) solve for $I(t)$ using the methods from the Complex Analysis course. If the initial current is zero, you should find

$$I(t) = \frac{\mathcal{E}_{battery}}{R}\left(1 - e^{-Rt/L}\right), \qquad (8.19)$$

showing that the current eventually reaches the value $I_s = \mathcal{E}_{battery}/R$, the value in the absence of the inductance.

Self-inductance L is a kind of electromagnetic inertia. The higher the value of L the longer the current takes to reach I_s. What would happen if the resistance were zero? Would $I_s \to \infty$? Yes (!), but only after an infinite time. If $R = 0$ then we would have to solve $\mathcal{E}_{battery} = L\,dI/dt$, and so $I = \mathcal{E}_{battery}t/L$ (see if you can confirm this result from Eq. (8.19) in the limit $R \to 0$). The current that increases linearly in time provides a *constant* back emf $\mathcal{E}_{ind}$ that *exactly* opposes the drive voltage $\mathcal{E}_{battery}$. Taking $R = 0$ also gives us the opportunity to understand that an inductor stores electromagnetic energy. The power supplied by the battery ($= \mathcal{E}_{battery}I$) can be equated to $I\,(L\,dI/dt)$, so that the energy stored by the inductor (W_L) as the current increases from 0 to $I = I(t)$ is just

$$W_L = \int_0^I IL\frac{dI}{dt}dt = \frac{1}{2}LI^2. \qquad (8.20)$$

With the resistance reinstated, this stored energy will saturate to $\frac{1}{2}LI_s^2$. This formula is rather analogous to the formula for the energy stored in a capacitor, $W_C = \frac{1}{2}CV^2$.

As a simple example, let us calculate the self-inductance of a long solenoid of length ℓ. We saw earlier (see Eq. (7.52)) that the magnetic field inside a solenoid of N turns per unit length is just $B = \mu_0 NI$. If the turns are circular with radius $a \ll \ell$ then *total* area linked by the flux is $\pi a^2 N\ell$. The total flux is therefore $\Phi_B = \pi a^2 N\ell B = \mu_0\pi a^2 N^2\ell I$. Hence

$$\mathcal{E}_{ind} = -\frac{d\Phi_B}{dt} = -\frac{d\left(\mu_0 N^2\ell\pi a^2 I\right)}{dt} = -\mu_0\pi a^2 N^2\ell\frac{dI}{dt}. \qquad (8.21)$$

The self-inductance of a long solenoid is $L = \mu_0\pi a^2 N^2\ell$.

8.5.1 *Problems*

(1) A coaxial cable consists of a solid cylindrical wire core of radius a carrying current I, and an outer hollow conducting cylinder (radius b) carrying a current $-I$. Show that the self-inductance per unit length of the coaxial cable, is $(\mu_0/2\pi)\ln(b/a)$.

(2) A flat loop of wire, having 50 turns and of area 0.02 m^2, is placed with its plane horizontal. At $t = 0$ a uniform magnetic field directed vertically and of magnitude of magnitude $B = 0.012\left(1 - e^{-2.5t}\right)$ T is switched on. (a) sketch how the induced emf ($\mathcal{E}(t)$) varies with time. (b) On a sketch of the loop indicate clearly the direction of the induced current. (c) Determine the value of the induced emf at $t = 0.5$ s. (d) If the resistance of the loop is 15 Ω what is the current in the loop at $t = 0.5$ s?

(3) Estimate the magnitude of the emf induced between the wing tips of an aircraft which is travelling southward at 300 $m.s^{-1}$ on a horizontal flight path in a region where the vertical component of the Earth's magnetic field is 18 μT. Neglect air resistance.

Chapter 9

Maxwell's Equations

We are now nearly ready to assemble the classical theory of electromagnetism. First, let us review the key equations that we have learned so far:

Gauss' law $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{9.1}$$

No magnetic monopoles $$\nabla \cdot \mathbf{B} = 0, \tag{9.2}$$

Ampère's magnetostatic law $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{9.3}$$

Faraday's law $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{9.4}$$

These equations (first given as Eqs. (3.4), (7.32), (7.43) and (8.15)) are *almost* Maxwell's equations.

9.1 A Missing Term

You may have noticed that I left out the equation of charge conservation from the above list , i.e.

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{9.5}$$

In fact, by examining the above list more carefully, we see by taking the divergence of Ampère's law (Eq. (9.3)) that something must be wrong. The divergence of the curl of anything is zero, whereas Eq. (9.3) suggests rather that $\nabla \cdot \mathbf{J} = 0$, which by the above charge conservation equation we know is not the case. The flux of **J** over a closed surface is compensated for by the loss of charge within the enclosed volume. This suggests that there is a term missing from Eq. (9.3). We know that Eq. (9.3) is the correct law for static fields, and so one is led to the idea that the missing term must relate to how the fields and charges vary in time. To fix matters we need to add a term to the right hand side of Eq. (9.3) whose divergence is $\mu_0 \partial\rho/\partial t$. A clue comes from Gauss' law, Eq. (9.1), from which we see that a very good candidate for our missing term is

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Taking the divergence of this will, by Eq. (9.1), yield precisely $\mu_0 \partial\rho/\partial t$. So it looks like the modified form of Ampère's law, applicable when the electric field varies with time is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{9.6}$$

If we are right (and ultimately only experiments can tell), then the Eq. (9.5) becomes a *consequence* of Maxwell's equations.

Another thought experiment leads us to a similar conclusion.[1] Fig. 9.1(a) shows a capacitor being charged up by a constant current. By applying the integral form for Ampère's law around the perimeter of area S_1 as shown, we see that $2\pi r B = \mu_0 I$. But now consider Fig. 9.1(b). This is exactly the same situation except that instead of having a plane surface S_2 enclosed by the circle we have distorted the surface to S_2 which now passes between the plates of the capacitor. Now we know that the important quantity for applying Ampère's law is the current passing through the surface, not the details of the surface itself. However, it appears that whereas there is a current flowing through surface S_1, there is no *physical* current flowing through S_2. The field between the plates of the capacitor is just $Q/(\epsilon_0 A)$ (see section 5.1). Therefore, as the plates are charging, the change in the magnitude of the electric field between them over time dt is just

$$dE = \frac{dQ/A}{\epsilon_0} = \frac{I dt}{A\epsilon_0}. \tag{9.7}$$

All will be well if we now form a current density $\mathbf{J}_d$ as

$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{I\hat{\mathbf{n}}}{A}. \tag{9.8}$$

[1] This was Maxwell's brilliant thought experiment.

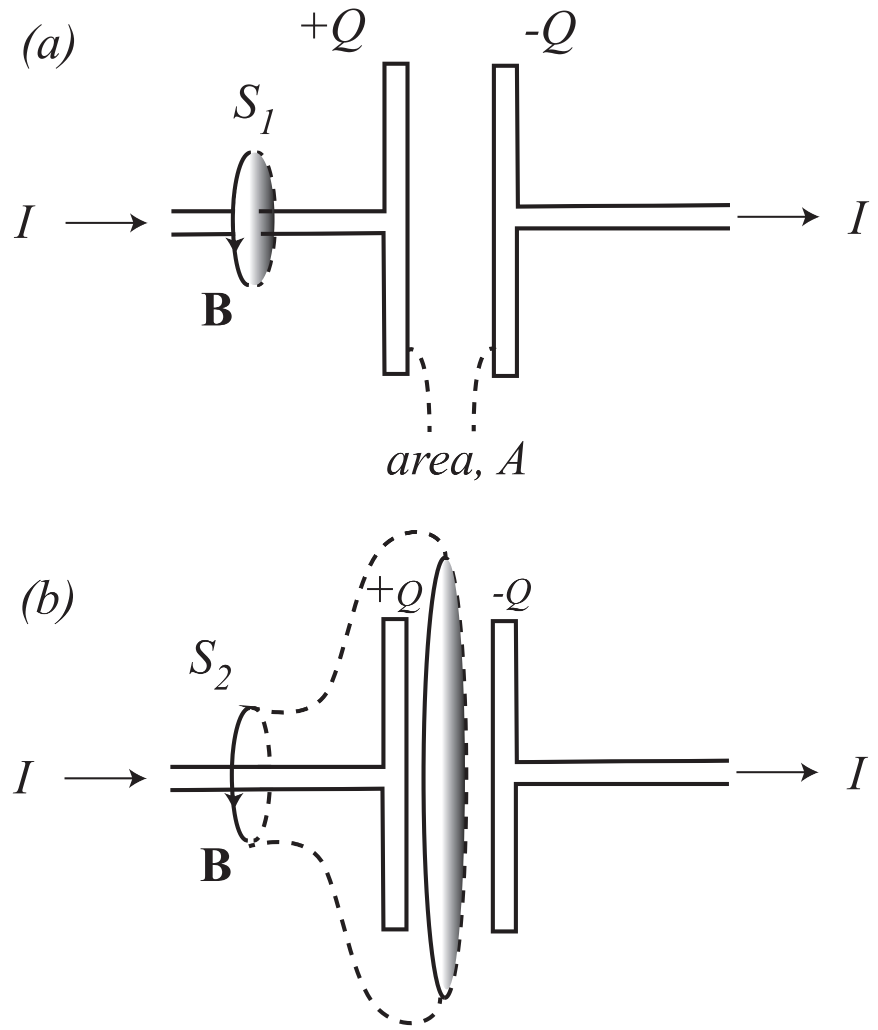

Fig. 9.1 Applying Ampère's law to charging a capacitor

If we add this current density to the real current density already present in Eq. (9.3), then we are again led to

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}.$$

In fact $\epsilon_0 \partial \mathbf{E}/\partial t$ is just the *displacement current* discussed earlier in the course. It is a form of current that can flow through vacuum!

9.2 Maxwell's Equations and Electromagnetic Waves

We now have the complete set of Maxwell's equations as

$$\textit{Gauss' law} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{9.9}$$

$$\textit{No magnetic monopoles} \qquad \nabla \cdot \mathbf{B} = 0, \tag{9.10}$$

$$\textit{Ampère's law} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{9.11}$$

$$\textit{Faraday's law} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{9.12}$$

This is a great moment where we now have the complete description of classical electromagnetism. You will learn a lot more about these equations in your later studies. Some of you (like me!) might even make a career out of trying to understand them.

For now we will content ourselves with our last grand finale - a simple calculation that leads us (as it did Maxwell, over 150 years ago) to the great idea that the above equations describe the propagation of light in free space.

Taking the curl of Eq. (9.12) and using a vector identity we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} + \nabla (\nabla \cdot \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}. \tag{9.13}$$

Now in free space, where there are no charges, so $\nabla \cdot \mathbf{E} = 0$ (by Eq. (9.9)). Also, since $\mathbf{J} = \mathbf{0}$ we can replace $\nabla \times \mathbf{B}$ with $\epsilon_0 \mu_0 \partial \mathbf{E}/\partial t$ (by Eq. (9.11)). Hence Eq. (9.13) can be written as

$$\nabla^2 \mathbf{E} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{9.14}$$

As an exercise, you might also care to show that a similar manipulation of the free-space Maxwell equations leads to

$$\nabla^2 \mathbf{B} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \tag{9.15}$$

Now you should recognise Eqs. (9.14) and (9.15) as *wave equations*. If the idea of a vector wave equation is a bit unfamiliar, then just write out the component equations and note that each equation represents a wave equation for each component. Also, if a wave equation in 3-D is unfamiliar, just compare with the one-dimensional wave equation for some scalar quantity f:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0, \tag{9.16}$$

where c is the speed of the wave. By comparing these equations we are led to the fantastic idea that Maxwell's equations *predict* the existence of *electromagnetic waves* that travel at the speed:

$$c = (\epsilon_0 \mu_0)^{-1/2}. \tag{9.17}$$

In principle, the constants $\epsilon_0 = 8.85418782 \times 10^{-12}$ F $\cdot$ m^{-1} and $\mu_0 = 1.25663706 \times 10^{-6}$ N $\cdot$ A^{-2} are determined by the experiments used to establish Gauss' law and Ampère's law. So the speed of these 'Maxwell waves' is *predicted to be*

$$c = \frac{1}{(8.85418782 \times 10^{-12} \times 1.25663706 \times 10^{-6})^{1/2}} = 2.9979 \times 10^{8}\ \text{m} \cdot \text{s}^{-1}. \tag{9.18}$$

Now the speed of light was measured *independently* (by Foucault in 1862) to be 2.98×10^8 m $\cdot$ s^{-1}. Maxwell, in the same year said

> We can scarcely avoid the conclusion that light consists in transverse undulations ... which is the cause of electric and magnetic phenomena.

which in my opinion is one of the most remarkable and far-reaching scientific conclusions of all time.