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# Relativity - Problem Sheet 1

**Topics covered: postulates of Relativity, proper time, time dilation, length contraction.**

**Questions to try in your own time**

1. It is common practice to scale velocities by the speed of light, so we define $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. $\beta$ is thus a dimensionless number between $-1$ and 1.

   (a) For what positive values of $\beta$ is $\gamma = 1.1$? $\gamma = 2$? $\gamma = 20$?

   (b) What values of $\gamma$ are given by $\beta = 0.09$? $\beta = 0.90$? $\beta = 0.99$?

   (c) For small values of $\beta$, derive an approximate expression for $\gamma$ in terms of $\beta$, keeping terms up to $\beta^2$. (Hint: use the binomial expansion.) How close is this approximation to the true value of $\gamma$ for $\beta = 0.09$?

2. A spaceship travelling at constant $\beta = 0.75$ travels from the Earth to the centre of the Milky Way, which in the Earth's inertial frame is a distance of $L = 27,000$ light-years away. (Note that 1 light-year is the distance light travels in 1 year.) Ignore any movement of the Earth relative to the centre of the Milky Way. It would be good practise to work in units of years and light-years directly, without converting to SI and back.

   (a) How long does the trip take as measured by an observer in the Earth's frame?

   (b) What is the distance $L'$ between Earth and the centre of the Milky Way as seen by an observer in the spaceship's inertial reference frame?

   (c) At what speed does the observer on the spaceship see the centre of the Milky Way coming towards the spaceship?

   (d) Given (b) and (c), what length of time does the observer on the spaceship measure for the trip? Is this consistent your answer to (a) given time dilation?

3. This question takes you through an observation which provides a direct experimental confirmation of time dilation and length contraction.

   When cosmic rays strike the upper atmosphere 10 km from the Earth's surface, they create muons. Muons decay quickly to electrons, with a lifetime of $\tau = 2.2\,\mu$s when they are at rest. Like nuclei, particles such as muons decay on average with an exponential function $N = N_0 e^{-t/\tau}$, where $N_0$ is the initial number of particles and $N$ is the number of particles that have not yet decayed after time $t$. Note that this means that if 1000 muons are created simultaneously, after $2.2\,\mu$s on average $1000e^{-1} = 368$ muons will left, while the rest will have decayed into electrons.

   (a) The muons move at a speed close to the speed of light: $\beta = 0.995$. Ignoring any time dilation, if one particular muon decayed after exactly 1 lifetime how far would it have travelled towards the Earth's surface before it decayed into an electron?

   (b) Again ignoring time dilation, use the exponential function of particle decay to find the average fraction of muons predicted to reach the Earth's surface.

   (c) We actually observe a much higher fraction of the muons created at the top of the atmosphere make it all the way to the Earth's surface and this is due to relativistic effects. Let's look at this in the muon's rest frame: for the muon the path length through the atmosphere to the surface is length contracted. How far does the muon measure the Earth's surface to be from the point at which it is created?

   (d) In the Earth's frame, we can explain the same effect by time dilation, as the muon lifetime can be treated as a physical clock. What is the lifetime of the moving muon in the Earth's frame?

   (e) Using the exponential function of particle decay, and the lifetime observed in Earth's frame, what percentage of muons is predicted to reach the Earth's surface now?

4. **Tutorial problem: length contraction**

In Lecture 3 we derived the length contraction formula $l' = l/\gamma$ by considering the measurement of a moving rod in two frames. This problem leads you through an alternate derivation based on the invariance of the speed of light. This is the inverse problem to the one in the same lecture, where we checked time dilation by assuming length contraction.

Consider a light clock with the light pulse emitter/detector and mirror separated in the $x$ direction by a distance $d$ in the clock rest frame, C. The time required for a light pulse to travel from the emitter to the mirror is $t_1 = d/c$ and this is clearly equal to the time for the return trip $t_2 = d/c$. Hence, the total period of the clock is

$$T = t_1 + t_2 = \frac{2d}{c}$$

Inertial frame M observes the entire apparatus moving to the right with speed $v$ in the $x$ direction. Frame M measures the separation to be $d'$, which we want to determine in terms of $d$ and $v$.

(a) View the situation from within frame M. Call the time for the light to travel from the emitter to the mirror in this frame $t_1'$. How far does the mirror move in this time? What distance does the light pulse travel in terms of $t_1'$ , $v$, and $d'$? This distance must equal $ct_1'$. You should therefore find that

$$t_1' = \frac{d'}{c - v}$$

(b) Using the same logic, find the time $t_2'$ for the light pulse to return from the mirror to the detector in frame M in terms of $d'$, $c$, and $v$. Hence find the clock period $T' = t_1' + t_2'$ in this frame.

(c) Which reference frame measures proper time for the light clock?

(d) We know that the periods in the two frames are related by the time dilation formula $T' = \gamma T$. Rewrite this equation in terms of $d'$, $d$, $v$, and $c$. You should now be able to retrieve the length contraction formula.

(e) Show that

$$t_1' \neq \gamma t_1 \qquad \text{and} \qquad t_2' \neq \gamma t_2$$

Why do these not hold?

**Multiple choice questions for coursework**

1. True or false: "In principle, it is possible for an observer following a pulse of light at a constant high speed to observe the light pulse to be almost stationary."

   (a) True

   (b) False

[2 marks]

2. Abi is in a spaceship moving at high speed relative to Ben, who is standing on an asteroid (a rock floating in space). She flies past him so that at $t = 0$, she is momentarily adjacent to Ben. At the instant that Abi's spaceship passes Ben, she sends two light pulses to him from her spaceship. If the light pulses are emitted a nanosecond ($10^{-9}$ seconds) apart according to Abi's clock, what will be the time interval between the pulses according to Ben?

   (a) Greater than one nanosecond

   (b) Equal to one nanosecond

   (c) Less than one nanosecond

[2 marks]

3. Also while Abi's spaceship passes Ben, Ben sends two light pulses to Abi. If Ben sends the light pulses a nanosecond apart according to his clock, what will be the time interval between the pulses according to Abi?

   (a) Greater than one nanosecond

   (b) Equal to one nanosecond

   (c) Less than one nanosecond

[3 marks]

4. Two identical rockets are floating in space one behind the other, at rest relative to an observer. The observer instructs both rockets to fire their engines at exactly the same time and, as they are identical, they then have the same acceleration. After a pre-programmed time, both engines shut off and the rockets drift at some constant speed relative to the observer. Since their accelerations are identical, they have the same distance between them at all times as measured by the observer, including after the engines are turned off.


   As shown in the diagram, a thin piece of thread was tied between the tail fin of the first and second rocket before the engines started, such that it was taut. After the acceleration, when the rockets are drifting, what will the state of the thread be?

   (a) Still taut.

   (b) Snapped.

   (c) Slack.

[3 marks]