Electricity and Magnetism study handout 3 hints, answers and solutions to non-assessed problems

1. Can a normal contact force ever be conservative?

Think about the definitions of conservative and non-conservative forces and what they imply. A key thing to consider is whether or not a contact force can change the mechanical energy of a system.

2. Which of the discussions developed in sections 4 and 5 could apply to masses in gravitational fields? Which discussions couldn't? Is there such a thing as gravitational capacitance and if so how would it work?

Of course the main difference is that there aren't real negative masses that attract positive masses.

3. Why is the potential inside a conductor in equilibrium always zero? What do the equipotential lines look inside?

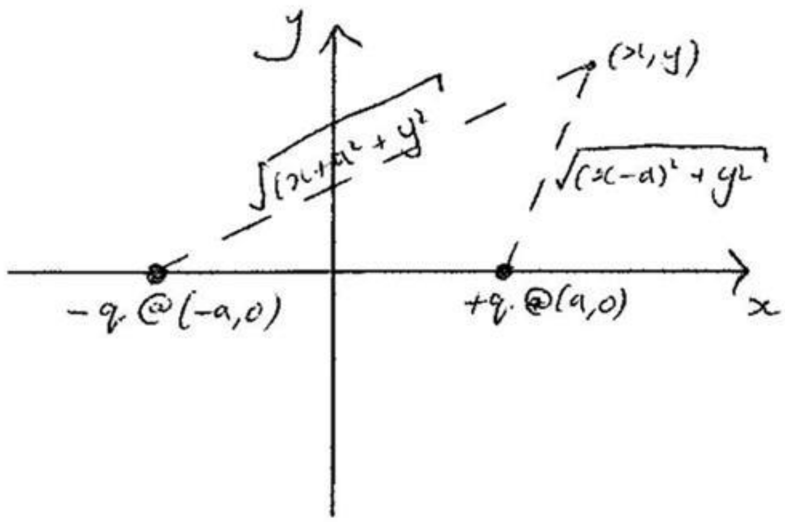
The answer this has been alluded to in lectures from both a force point of view and an energy point of view. As the potential inside a conducting material is zero would having any equipotential lines at all make much sense?

4. What are the equipotential lines for (a) an infinite line of charge? (b) an infinite sheet of charge?

(a) Concentric cylinders about the line; (b) infinite planes parallel to the sheet.

5. Calculate the field strength due to an electric dipole along the axis and perpendicular bisector using the potential method instead of forces.

Consider a general point at (x, y) in the vicinity of a dipole in the same positions as given in figure 2.6 in the lecture notes:



The potential at (x, y) is the sum of the potentials to each components of the dipole and using the distances given on the diagram is thus:

$$V = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(x-a)^2+y^2}} - \frac{1}{\sqrt{(x+a)^2+y^2}}\right\}$$

So using $E = -\nabla V = -\hat{\imath}\frac{\partial V}{\partial x} - \hat{\jmath}\frac{\partial V}{\partial y}$ in two dimensions, the electrostatic field at any general point (x, y) is given by:

$$E = \frac{q}{4\pi\epsilon_0}\left\{\hat{\imath}\left(\frac{x-a}{((x-a)^2+y^2)^{3/2}} - \frac{x+a}{((x+a)^2+y^2)^{3/2}}\right) + \hat{\jmath}\left(\frac{y}{((x-a)^2+y^2)^{3/2}} - \frac{y}{((x+a)^2+y^2)^{3/2}}\right)\right\}$$

where the derivatives are fiddly but have been obtained in the usual manner.

For the bisector we need to set $x = 0$ and approximate $y \gg a$ which gives $E \approx -\frac{2aq\hat{\imath}}{4\pi\epsilon_0 y^3}$ as required.

For the axis we need to set $y = 0$ and approximate $x \gg a$ which gives $E = \frac{q\hat{\imath}}{4\pi\epsilon_0}\left(\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right)$ which is exactly as in section 2.4 before the approximation is imposed as required.

6 Why does the capacitance of a coaxial cable become large as the inner radius becomes close to the outer radius?

Think about the definition capacitance and why a small separation means a large capacitance for a parallel plate capacitor.

7. A capacitor of $4\ \mu\text{F}$ is charged to $20\ \text{V}$ and then placed in series with another capacitor of $6\ \mu\text{F}$. Describe what happens and find the final charges on the plates and potential differences across each capacitor.

Basically the charged capacitor drives a current through the circuit until the potential difference across both capacitors reaches an equilibrium point. The charge imbalance on each side of the circuit remains the same.

The charged capacitor has a charge on each plate of $Q = CV = 4 \times 10^{-6} \times 20 = 80\ \mu\text{C}$ and this will eventually be shared between both capacitors. As this is conserved, and the final state it can be written that $80 \times 10^{-6} = C_1V + C_2V$ where V is the final steady state potential difference across each capacitor. Hence $V = \frac{80\times10^{-6}}{C_1+C_2} = \frac{80}{4+6} = 8\ \text{V}$.

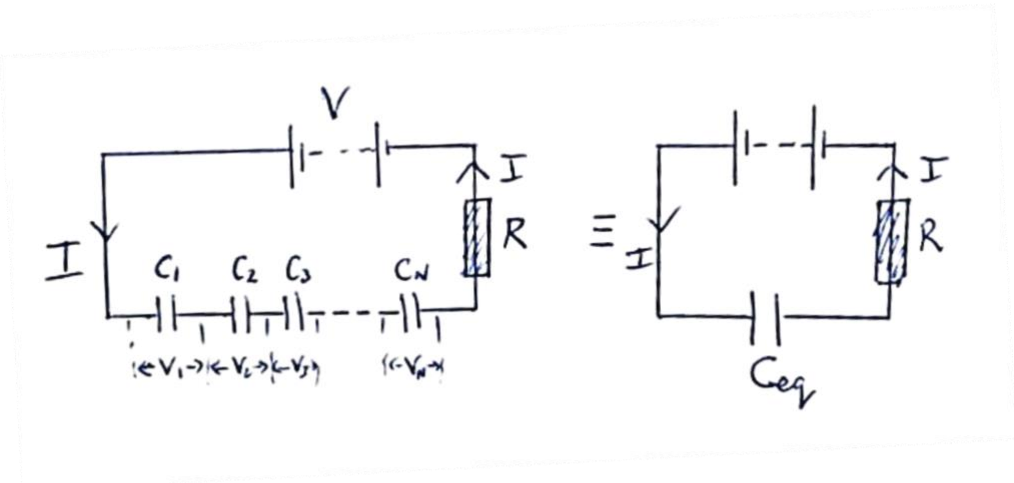
This gives a charge of the capacitors of $4 \times 10^{-6} \times 8 = 32\ \mu\text{C}$ and $6 \times 10^{-6} \times 8 = 48\ \mu\text{C}$.

Note that the conservation of energy could not be used here as energy is reduced during the equilibrating process (by $480\ \mu\text{J}$ in fact if you calculate the energy stored before and after). This is because the circuit always has some resistance in the wires and a current flowing dissipates energy - this is ultimately the definition of resistance. It doesn't matter what the value of the resistance is, the same amount of energy is always dissipated.

8. By considering the PD across the plates and the current through the circuit, when a battery is connect across them, find the equivalent capacitance of (a) N capacitors in series (b) N capacitors in parallel.

<u>Series</u>

Consider a potential difference, V, across N capacitors in series and the equivalent circuit made of just one capacitor:



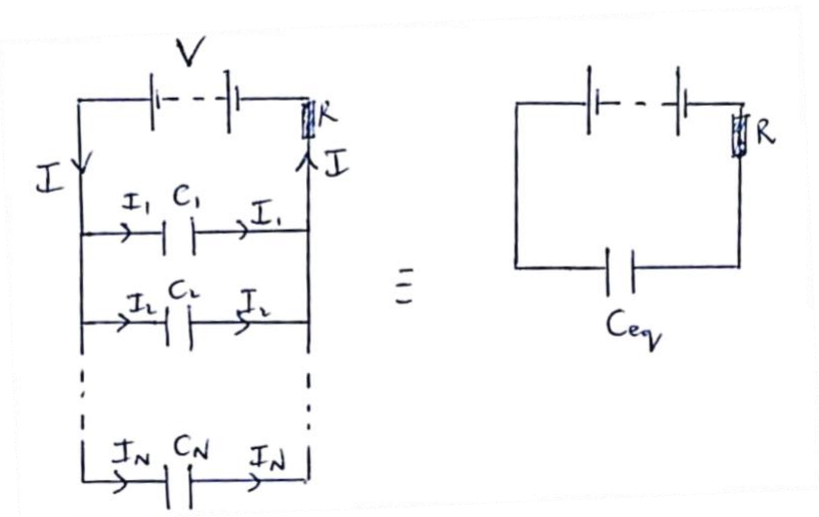
In the series circuit each of the capacitors has a PD across it with the total PD being given by $V = V_1 + V_2 + \cdots + V_N$.

Using $Q = CV$ for each capacitor this can be written as $\frac{Q_{eq}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \cdots + \frac{Q_N}{C_N}$ where Q_{eq} is the charge on the equivalent capacitance of value C_{eq}.

Taking time derivatives of both sides of the equation gives $\frac{I_{eq}}{C_{eq}} = \frac{I_1}{C_1} + \frac{I_2}{C_2} + \cdots + \frac{I_N}{C_N}$ as current is rate of flow of charge. As the current through each capacitor in series is the same and identical to that of the total current in the circuit this simplifies to $\frac{I}{C_{eq}} = \frac{I}{C_1} + \frac{I}{C_2} + \cdots + \frac{I}{C_N}$ which further simplifies to the final answer, $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$ i.e. the equivalent capacitance of several capacitors in series is given the reciprocal of the sum of the reciprocals of each individual capacitor.

<u>Parallel</u>

Now consider N capacitors in parallel:



In this case the currents add linearly i.e. $I = I_1 + I_2 + \cdots I_N$ and thus the sum of the charges is equal to the charge on the equivalent unit i.e. $Q_{eq} = Q_1 + Q_2 + \cdots Q_N$

Using $Q = CV$ this gives $C_{eq}V = C_1V + C_2V + \cdots C_NV$ as the PD is the same across each capacitor. This therefore simplifies to $C_{eq} = C_1 + C_2 + \cdots C_N$ which is the intuitively simple final result i.e. the equivalent capacitance of capacitors in parallel if the sum of each capacitor.

9. A potential difference of magnitude V_0 is placed across a resistor R is series with a capacitor C. Find the following as a function of time: (a) the PD across the plates of the capacitor, (b) the charge on the plates of the capacitor, (c) the current flowing through the circuit.

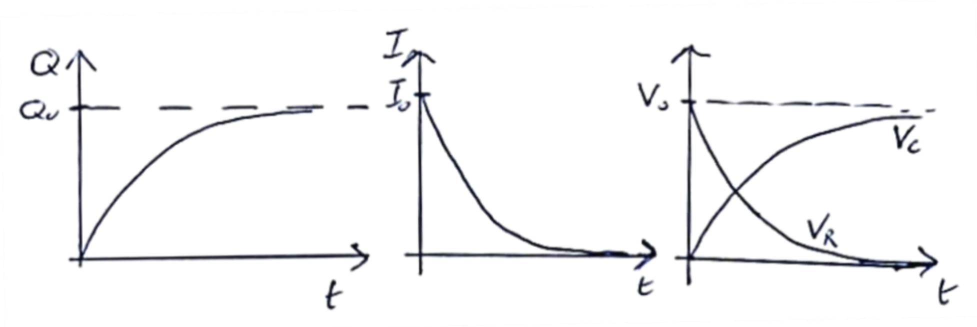
If the capacitor and the resistor are in series the sum of the PDs across each will equal the total PD so using $V = IR$ and $Q = CV$ this gives $V_0 = IR + \frac{Q}{C}$. When the circuit is initially switched on the charge on the capacitor is zero so the PD across the capacitor is zero and the PD across the resistor is V_0. As the charge on the capacitor builds up, the PD across it becomes greater, the current falls and eventually all the PD is across the capacitor.

As the current through the resistor can be written $\frac{dQ}{dt}$ we get a first order differential equation $V_0 = \frac{dQ}{dt}R + \frac{Q}{C}$ which can be rewritten $CR\frac{dQ}{dt} = CV_0 - Q$ which leads to the integral $CR\int_{Q'=0}^{Q'=Q_0} \frac{dQ'}{CV_0 - Q'} = \int_{t'=0}^{t'=t} dt$ where Q_0 is the final charge on the capacitor.

Solving and rearranging leads to $Q = CV_0\left(1 - e^{-t/_{CR}}\right) = Q_0\left(1 - e^{-t/_{CR}}\right)$.

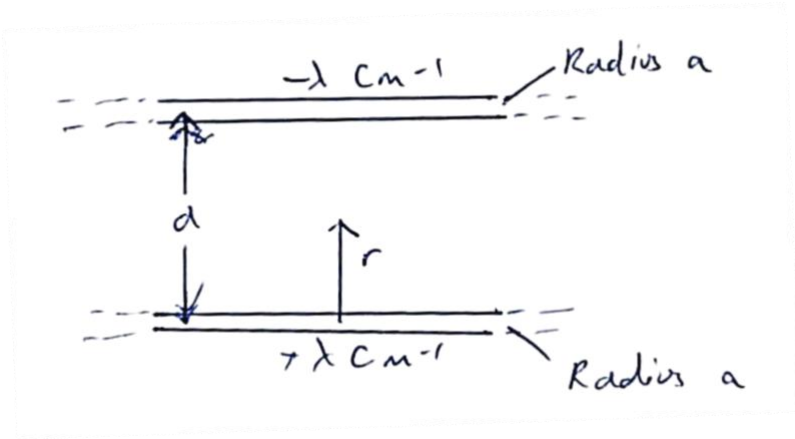
The current can be found by differentiating and is $I = \frac{Q_0}{CR}e^{-t/_{CR}} = I_0 e^{-t/_{CR}}$.

The PD across the resistor is therefore $V_R = Ve^{-t/_{CR}}{}_0$ and the PD across the capacitor is $V_C = V_0\left(1 - e^{-t/_{CR}}\right)$.



10. Find an expression for the capacitance per unit length of two parallel, infinitely long straight wires of radius a and separation $d$for the case when $d \gg a$.

Consider the arrangement below:



The electric field strength at a general position $\boldsymbol{r}$ from the centre of one wire using the principle of superposition and the result from result from problem 1.2a from section 2 is $\frac{\lambda}{2\pi\epsilon_0 r} + \frac{\lambda}{2\pi\epsilon_0(d-r)}$.

So the potential difference from the inner edge of one wire to the other is $\int \boldsymbol{E} \cdot \boldsymbol{dr} = \frac{\lambda}{2\pi\epsilon_0}\int_a^{d-a}\left(\frac{1}{r} + \frac{1}{d-r}\right)\boldsymbol{dr}$.

This solves and rearranges to yield $\frac{\lambda}{\pi\epsilon_0}\ln\frac{d-a}{a}$ which, in the limit that $\boldsymbol{d} \gg \boldsymbol{a}$ as specified gives a potential difference between the wires of $\sim\frac{\lambda}{\pi\epsilon_0}\ln\frac{d}{a}$

So using $\boldsymbol{Q} = \boldsymbol{CV}$ the capacitance per unit length of the arrangement is $\frac{\pi\epsilon_0}{\ln{}^{d}/_{a}}$.