Vibrations and Waves — Problem Sheet 1 ANSWERS

1. (i) $\text{Re}[\tilde{x}] = 2\cos(6t)$

   (ii) $\text{Re}[\tilde{x}] = -3\sin(5t)$

   (iii) $\text{Re}[\tilde{x}] = 2\cos(6t) - 3\sin(6t)$

   (iv) $\text{Re}[\tilde{x}] = \cos(6t) + 5\sin(6t)$

2. (i) Since force is same on each spring $F = -k_1 x_1 = -k_2 x_2 \Rightarrow x_1 = k_2 x_2 / k_1$ as required.

   (ii) The force applied to the combined system is F and resulting combined extension is $x_1 + x_2$. Substituting for x_1 and x_2 from (i) into the equation given for k_{eff} gives $F = -k_{\text{eff}}(-F/k_1 - F/k_2) \Rightarrow k_{\text{eff}} = (1/k_1 + 1/k_2)^{-1}$.

   (iii) For springs in parallel, the extension is the same for both springs ($x_1 = x_2 = x$). The total force is the sum of the forces due to each spring, $F = F_1 + F_2 = -k_1 x + k_2 x = -(k_1 + k_2)x$. Therefore the effective spring constant is $k_{\text{eff}} = k_1 + k_2$.

3. (i) $A = 0.05\,\text{m}$

   (ii) $\omega_0 = 7.51\text{s}^{-1}$

   (iii) $f = \omega_0/(2\pi) = 1.20\,\text{Hz}$

   (iv) $T = 1/f = 0.84\,\text{s}$

   Using $\omega_0^2 = k/m$ gives $k = m\omega_0^2 = (0.1)(7.51)^2\text{N/m} = 5.64\,\text{N/m}$.

   Initial extension x_0 is such that spring restoring force balances gravitational force, so that $-kx = -mg$. Then $x_0 = F/k = mg/k = g/\omega_0^2 = (9.8\,\text{m/s}^2)/(7.51\,/\text{s})^2 = 0.17\,\text{m}$.

4. $v(t) = \text{d}x(t)/\text{d}t = -4A\sin(4t + \phi)$

   Use $A = \sqrt{x^2 + (\dot{x}/\omega)^2}$ and $\cos\phi = x/A$ and $\sin\phi = -\dot{x}/(\omega A)$

   (i) $A = \sqrt{0.3^2 + (0)^2} = 0.3\,\text{m}$ and $\cos\phi = 1, \sin\phi = 0 \quad \Rightarrow \quad \phi = 0$

   (ii) $A = \sqrt{(-0.5)^2 + (0)^2} = 0.5\,\text{m}$ and $\cos\phi = -1, \sin\phi = 0 \quad \Rightarrow \quad \phi = \pi$

   (iii) $A = \sqrt{(0)^2 + (1.2/4)^2} = 0.3\,\text{m}$ and $\cos\phi = 0, \sin\phi = (-1.2)/(4 \times 0.3) = -1 \quad \Rightarrow \phi = -\pi/2$

5. (i) When the liquid is displaced by $= +x$ on the RHS (and $-x$ on the LHS) there is a net gravitational restoring force on the liquid equivalent to the weight difference on the two sides. $F = -\text{volume} \times \text{density} \times g = -2xA\rho g$ (minus since it's a restoring force). Using $F = ma$, with the total mass of the liquid given by $LA\rho$, this gives

   $$-2xA\rho g = LA\rho \frac{\text{d}^2 x}{\text{d}t^2}$$

   (ii) Rearranging (and cancelling $A\rho$ from both sides) gives

   $$\frac{\text{d}^2 x}{\text{d}t^2} + \frac{2g}{L}x = 0$$

   which is the same form as the SHM equation $\ddot{x} + \omega_0^2 x = 0$ and therefore has solutions $x(t) = B\cos(\omega_0 t + \phi)$ with $\omega_0 = \sqrt{2g/L}$. The amplitude B is determined by initial conditions $x(0) = h, \dot{x}(0) = 0$ which gives $B = h, \phi = 0$ giving $x(t) = h\cos(\omega_0 t)$.

(iii) $\omega_0 = \sqrt{2g/L}$.

(iv) $v(t) = \dot{x}(t) = -\omega_0 h \sin(\omega_0 t)$

(v) $a(t) = \ddot{x}(t) = -\omega_0^2 h \cos(\omega_0 t)$

(vi) From the diagram, the change in PE from equilibrium is equivalent to moving volume Ax of liquid (from the left to the right side) and raising it a distance x. Therefore PE = $(\rho Ax)gx = g\rho Ax^2 = g\rho Ah^2 \cos^2(\omega_0 t)$

(vii) KE = $(1/2)m\dot{x}^2 = (1/2)\rho LA\omega_0^2 h^2 \sin^2(\omega_0 t)$. But $\omega_0^2 = 2g/L$, and so KE = $\rho gAh^2 \sin^2(\omega_0 t)$.

(viii) Total energy TE = PE + KE = ρgAh^2.

(ix) KE = TE − PE = $\rho gAh^2 - PE = \rho gAh^2 - \rho gAx^2 = \rho gA(h^2 - x^2)$