Vibrations and Waves — Problem Sheet 4

1. Which of the following waves (written in vector form, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are unit vectors in the x, y and z directions) are transverse waves and which are longitudinal, which are travelling waves and which are standing waves and in which direction are the travelling waves propagating?

   i) $\boldsymbol{\psi}(z,t) = A\hat{\mathbf{x}}\cos(\omega t - kz)$

   ii) $\boldsymbol{\psi}(z,t) = A\hat{\mathbf{z}}\cos(\omega t)\sin(kz)$

   iii) $\boldsymbol{\psi}(x,t) = A\hat{\mathbf{x}}\cos(\omega t + kx)$

   iv) $\boldsymbol{\psi}(z,t) = A_1\hat{\mathbf{x}}\cos(\omega t - kz) + A_2\hat{\mathbf{z}}\sin(\omega t - kz)$

   v) $\boldsymbol{\psi}(z,t) = A_1\hat{\mathbf{x}}\cos(\omega t - kz) + A_2\hat{\mathbf{z}}\cos(\omega t + kz)$

   vi) $\boldsymbol{\psi}(z,t) = A\hat{\mathbf{x}}\cos(\omega t - kz) + A\hat{\mathbf{x}}\cos(\omega t + kz)$

2. The following two waves in a medium are superposed:

$$\psi_1(z,t) = A\cos(10t - 5z)$$
$$\psi_2(z,t) = A\cos(9t - 4z)$$

   where z is in metres and t in seconds.

   i) Write an equation for the combined disturbance.

   ii) What is the beat frequency in Hz?

   iii) What is its group velocity?

3. This question tests your understanding of phase and group velocities.

   i) Using the chain rule for differentiation, show that $\mathrm{d}\omega^2/\,\mathrm{d}(k^2) = v_\mathrm{g}v_\mathrm{p}$ where ω, k are the frequency and wavevector for a wave, and v_g and v_p are the group and phase velocities, respectively.

   ii) Electromagnetic waves propagate through plasmas with a refractive index given by $n = \sqrt{1 - \omega_\mathrm{p}^2/\omega^2}$ where ω_p is a constant for given plasma conditions known as the *plasma frequency*. Write down an expression for the phase velocity for waves in a plasma and hence show that ω and k are related by the *plasma dispersion* relation $\omega^2 = c^2k^2 + \omega_\mathrm{p}^2$, where c is the speed of light.

   iii) Using (i) show that this dispersion relation leads to the result that $v_\mathrm{g} = c\sqrt{1 - \omega_\mathrm{p}^2/\omega^2}$.

   iv) Can v_p exceed c? What about v_g? Comment on your answers.

4. Short wavelength ripples ($\lambda \lesssim 1$ cm) moving on the surface of water are controlled by surface tension. The phase velocity of such ripples is given by

$$v_\mathrm{p} = \sqrt{\frac{2\pi S}{\rho\lambda}}$$

   where S is the surface tension and ρ the density of water.

   i) Use dimensional analysis to show that S has units of force per unit length.

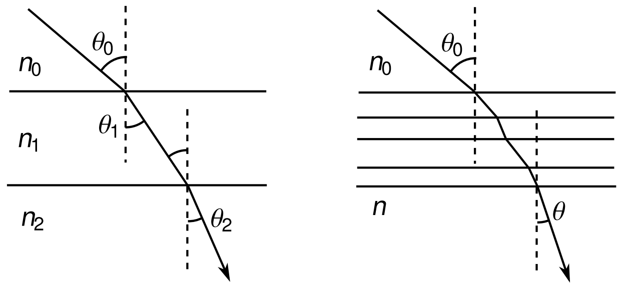
ii) Determine the dispersion relation and sketch it as a function of wavenumber k. Is the dispersion *anomalous* or *normal*. Explain your answer.

iii) Show that the group velocity of a wavegroup of ripples is 1.5 times the phase velocity.

iv) A pebble is tossed into a pond of 2 metres diameter at one side. This creates a ripple wavegroup consisting of waves with λ in the range 0.1–1 cm. How long does it take for the wavegroup to cross the pond? For water $S = 0.073\,\mathrm{N/m}$ (at 20°) and $\rho = 10^3\,\mathrm{kg/m^3}$.

5. *** The speed (phase velocity) of bending waves in an isotropic plate (2-D) is given by $v = C\omega^{1/2}$ where C is a constant and ω the angular frequency. Derive the dispersion relation. What type of dispersion is this? Derive a formula for the frequencies of the normal modes on a rectangular plate (assume closed boundary conditions). A rectangular plate with dimensions $0.6\,\mathrm{m} \times 0.4\,\mathrm{m}$ has a fundamental frequency of 300 Hz. Calculate the constant C. What are the frequencies of the next 5 modes in order of frequency? (You might find these calculations easier on a computer.)

6. This question will test you ability to apply Snell's law of refraction. Light is incident at an angle θ_0 from a medium with refractive index n_0 onto one with refractive index n_1 and then onto one with refractive index n_2 (left diagram). The interfaces between the media are parallel.



i) Use Snell's law to obtain an expression for the angle θ_2.

ii) Hence show that the angle in the bottom layer of a stack of parallel layers of different refractive indices does not depend on the intermediate layers (right diagram).

iii) The refractive index of air at sea level is $n = 1.0003$. Obtain an expression for the apparent angle θ of the Sun from zenith (i.e., from the vertical) at sea level when its true angle is θ_0 .

iv) Plot the deviation $\theta - \theta_0$ as a function of θ_0.

v) Use your expression to obtain an estimate of the deviation of the Sun from its true angle when the Sun is on the horizon. Why is you result likely to be an overestimate?

7. The following is probably the most complicated situation you can get for the Doppler effect—moving source and echo from moving target!

A bat flying at speed u_B is out hunting. Using echo-location it detects a moth flying at speed u_M in a straight line *away* from it towards a tree. The bat emits a sound wave of frequency f_B towards the moth. The speed of sound in air is v. You may use $f' = (v + u_R)/(v - u_S)f$ for source (S) and receiver (R) moving with speeds u_R and u_S respectively *towards* each other relative to the medium. (The double dash in (c),(d) indicates that two Doppler shifts are involved.)

i) Show that the sound wave arriving at the tree has frequency $f'_T = [v/(v - u_B)]f_B$.

ii) Show that the sound wave arriving at the moth has frequency $f'_M = [(v - u_M)/(v - u_B)]f_B$.

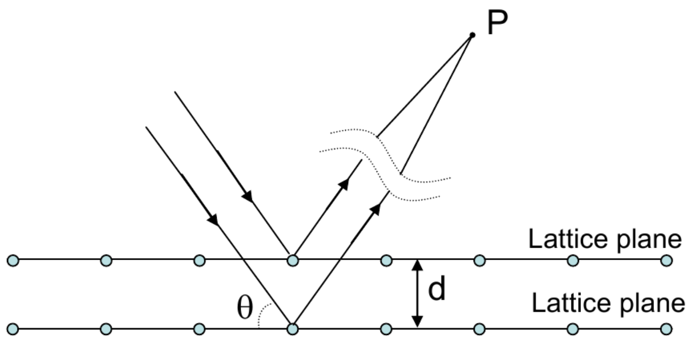
iii) Show that the echo received by the bat from the tree is $f''_T = [(v + u_B)/(v - u_B)]f_B$.

iv) Show that the echo received by the bat from the moth is

$$f''_M = \left(\frac{v + u_B}{v + u_M}\right)\left(\frac{v - u_M}{v - u_B}\right) f_B.$$

v) Some bats emit a burst of sound waves that changes its frequency from start to finish (it has a *frequency chirp*). If $u_B = 4\,\text{m/s}$, $u_M = 0.1\,\text{m/s}$ and $v = 344\,\text{m/s}$ what frequencies emitted by bat would result in echoes from the tree and the moth being received at 10 kHz by the bat.

8. Waves scattered from atoms in a crystal can interfere to make a “diffraction pattern”. This is known as “Bragg Diffraction”. The details of this diffraction pattern can be used to determine the crystal’s structure.



i) Using the figure above, show that the path difference between waves scattered from atoms in adjacent planes is $2d \sin\theta$. Note that it is customary to measure θ from the lattice plane and *not* relative to the normal to the surface.

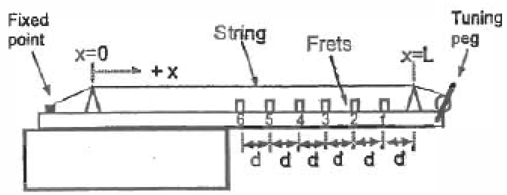
ii) What is the condition for constructive interference of waves at a (distant) point P on a detector?

iii) Lattice planes in crystals are typically spaced by a few Ångstroms (1 Å = 10^{-10} m). What kind of waves are diffracted by crystals?

The following questions are from a past exam paper (different lecturer). Note: The notation in places is slightly different from those used in the lecture course. [Numerical Answers: 8i)b) $T^* = 4000$ N, c) $f_{\text{beat}} = 111$ Hz. ii) b)3.44 m, c)0.254 mW/m^2]

SMVWQP Exam May 2004 V&W Long Question (40 minutes)

7. A guitar string is fixed at one end. The other end it is attached to a tuning peg which, when rotated, allows the tension T^* in the string to be adjusted. The part of the string which vibrates is between two ridges at $x = 0$ and $x = L$. Other lower ridges, called frets, on the guitar neck allow the musician to adjust the length of the part of the string which vibrates by pushing the string down on to the fret with a finger. Let the frets numbered from $m = 1$ to 6 be equally spaced along the guitar neck at intervals of d.



(i) (a) By considering the addition of two travelling transverse plain waves going in opposite directions along the string:

$$y_+(x,t) = A\cos(\omega t - kx + \phi)$$

$$y_-(x,t) = -A\cos(\omega t + kx + \phi)$$

show that the guitar string will support standing waves of wavelength:

$$\lambda_n = 2(L - md)/n$$

where n is a positive integer and m is the number of the fret where the musicians finger is placed ($m = 0$ if the string is not held down on any fret). [5 marks]

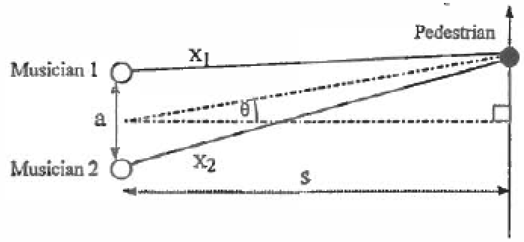
(b) The phase speed of a transverse wave on a string is given by:

$$v = \sqrt{T * /\sigma}$$

where σ is the mass per unit length of the string. If the musician wishes the fundamental frequency ($n = 1$) to be equal to 1 kHz when no frets are used ($m = 0$), what tension T^* needs to be applied to the string? Take $\sigma = 10^{-3}$ kgm^{-1} and $L = 1$ m. [2 marks]

(c) A second identical string on the same guitar is adjusted to the same T^*. The musician then plays the first and second string simultaneously with his finger on no fret ($m = 0$) on the first string and fret $m = 2$ on the second string. This produces beats. What is their frequency ν_{beat} if $d = 0.05$ m. [2 marks]

(ii) (a) The musician goes to the local town square and tries to make some money by playing his guitar. A second musician turns up and stands a distance a away from the first. They both tune their guitars by playing the same note at 1 kHz. A pedestrian walks parallel to them along the other side of the square a distance s away. As he walks the sound intensity from the guitars varies with his position.



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Show that the sound intensity will reach a maximum when:

$$\sin\theta_{max} = \frac{pv}{a\nu}$$

where p is an integer, v is the speed of sound in air and ν is the wave frequency. [3 marks]

(b) Using this, calculate the distance Δy the pedestrian walks between maximum points of sound intensity. Let $s = 50$ m, $a = 5$ m and $v = 344$ m s^{-1}. [3 marks]

(c) Each guitar emits an average power of 1 W. Assume that the sound wave moves out in a hemisphere (does not penetrate the ground). What is the maximum intensity the pedestrian will hear as he walks down the road? [3 marks]

[TOTAL 18 marks]