Paul Dauncey Available 18 May 2018

Relativity - Solutions to Problem Sheet 1

Topics covered: postulates of Relativity, proper time, time dilation, length contraction.

Questions to try in your own time

1. It is common practice to scale velocities by the speed of light, so we define $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. β is thus a dimensionless number between -1 and 1.

   (a) For what positive values of β is $\gamma = 1.1$? $\gamma = 2$? $\gamma = 20$?
   ANSWER: $\gamma = 1.1$ implies $1/\sqrt{1-\beta^2} = 1.1$, or $1-\beta^2 = 1/1.21$. Thus $\beta = 0.4166$. Similarly, for $\gamma = 2$, $\beta = 0.8660$ and for $\gamma = 20$, $\beta = 0.9987$.

   (b) What values of γ are given by $\beta = 0.09$? $\beta = 0.90$? $\beta = 0.99$?
   ANSWER: $\gamma = 1/\sqrt{1-\beta^2}$ so for $\beta = 0.09$ then $\gamma = 1/\sqrt{0.9919} = 1.004$. Similarly for $\beta = 0.90$ then $\gamma = 2.294$ and for $\beta = 0.99$ then $\gamma = 7.089$

   (c) For small values of β, derive an approximate expression for γ in terms of β, keeping terms up to β^2. (Hint: use the binomial expansion.) How close is this approximation to the true value of γ for $\beta = 0.09$?
   ANSWER: $\gamma = (1-\beta^2)^{-1/2}$. Using the binomial expansion $(1+x)^n \approx 1 + nx$, this gives

   $$\gamma \approx 1 + (-1/2)(-\beta^2) \approx 1 + \beta^2/2$$

   Using this approximation, then for $\beta = 0.09$, $\gamma \approx 1 + 0.0081/2 \approx 1.00405$, while the exact value is 1.00407 so they differ only at the sixth significant figure.

2. A spaceship travelling at constant $\beta = 0.75$ travels from the Earth to the centre of the Milky Way, which in the Earth's inertial frame is a distance of $L = 27,000$ light-years away. (Note that 1 light-year is the distance light travels in 1 year.) Ignore any movement of the Earth relative to the centre of the Milky Way. It would be good practise to work in units of years and light-years directly, without converting to SI and back.

   (a) How long does the trip take as measured by an observer in the Earth's frame?
   ANSWER: This requires no knowledge of Relativity; the time is distance over speed so for an observer on the Earth $t = 27,000c/0.75c = 36,000$ years.

   (b) What is the distance L' between Earth and the centre of the Milky Way as seen by an observer in the spaceship's inertial reference frame?
   ANSWER: The value of $\gamma = 1/\sqrt{1-0.75^2} = 1.51$. The rest frame of the length is the Earth-Milky Way frame, so due to length contraction the spaceship sees a shorter length of $L' = L/\gamma = 27,000/1.51 = 17900$ light-years.

   (c) At what speed does the observer on the spaceship see the centre of the Milky Way coming towards the spaceship?
   ANSWER: The relative speed of the Earth (and hence centre of the Milky Way) and the spaceship is $0.75c = 2.25 \times 10^8$ m/s, so the Earth sees the spaceship moving with this speed. Similarly the spaceship sees the Earth (and hence centre of the Milky Way) moving at the same speed, but in the opposite direction.

   (d) Given (b) and (c), what length of time does the observer on the spaceship measure for the trip? Is this consistent your answer to (a) given time dilation?
   ANSWER: Using the contracted length, the spaceship measures a time of $t' = L'/0.75 = 17900/0.75 = 23800$ years. The observer's clock on the spaceship records proper time, so $t' = t/\gamma = 36000/1.51 = 23800$ years. Hence, the results are consistent.

3. This question takes you through an observation which provides a direct experimental confirmation of time dilation and length contraction.

   When cosmic rays strike the upper atmosphere 10 km from the Earth's surface, they create muons. Muons decay quickly to electrons, with a lifetime of $\tau = 2.2\,\mu$s when they are at rest. Like nuclei, particles such as muons decay on average with an exponential function $N = N_0 e^{-t/\tau}$, where N_0 is the initial number of particles and N is the number of particles that have not yet decayed after time t. Note that this means that if 1000 muons are created simultaneously, after $2.2\,\mu$s on average $1000e^{-1} = 368$ muons will left, while the rest will have decayed into electrons.

(a) The muons move at a speed close to the speed of light: $\beta = 0.995$. Ignoring any time dilation, if one particular muon decayed after exactly 1 lifetime how far would it have travelled towards the Earth's surface before it decayed into an electron?
ANSWER: The distance it would travel would be $D = \beta c\tau = 0.995 \times 3 \times 10^8 \times 2.2 \times 10^{-6} = 657$ m.

(b) Again ignoring time dilation, use the exponential function of particle decay to find the average fraction of muons predicted to reach the Earth's surface.
ANSWER: The fraction that would reach the surface would be

$$N/N_0 = e^{-t/\tau} = e^{-\frac{10^4/(0.995\times3\times10^8)}{2.2\times10^{-6}}} = 2.4 \times 10^{-7}$$

(c) We actually observe a much higher fraction of the muons created at the top of the atmosphere make it all the way to the Earth's surface and this is due to relativistic effects. Let's look at this in the muon's rest frame: for the muon the path length through the atmosphere to the surface is length contracted. How far does the muon measure the Earth's surface to be from the point at which it is created?
ANSWER: For $\beta = 0.995$, then $\gamma = 10.01$. Hence, in the muon's frame: $L' = L/\gamma = 1.0$ km. This is 10 times shorter than in Earth's frame - we therefore expect significantly more muons to arrive at the Earth's surface.

(d) In the Earth's frame, we can explain the same effect by time dilation, as the muon lifetime can be treated as a physical clock. What is the lifetime of the moving muon in the Earth's frame?
ANSWER: In Earth's frame: $t_{\text{life}} = \gamma\tau = 2.2 \times 10^{-5}$ s. (The life time is 10 times longer in Earth's frame than in the muon's rest frame).

(e) Using the exponential function of particle decay, and the lifetime observed in Earth's frame, what percentage of muons is predicted to reach the Earth's surface now?
ANSWER: The predicted fraction to reach the Earth's surface is

$$N/N_0 = e^{-t/\gamma\tau} = e^{-\frac{10^4/(0.995\times3\times10^8)}{2.2\times10^{-5}}} = 0.22$$

This means 22% of the muons created in the upper atmosphere reach the Earth's surface.

4. **Tutorial problem: length contraction**

In Lecture 3 we derived the length contraction formula $l' = l/\gamma$ by considering the measurement of a moving rod in two frames. This problem leads you through an alternate derivation based on the invariance of the speed of light. This is the inverse problem to the one in the same lecture, where we checked time dilation by assuming length contraction.

Consider a light clock with the light pulse emitter/detector and mirror separated in the x direction by a distance d in the clock rest frame, C. The time required for a light pulse to travel from the emitter to the mirror is $t_1 = d/c$ and this is clearly equal to the time for the return trip $t_2 = d/c$. Hence, the total period of the clock is

$$T = t_1 + t_2 = \frac{2d}{c}$$

Inertial frame M observes the entire apparatus moving to the right with speed v in the x direction. Frame M measures the separation to be d', which we want to determine in terms of d and v.

(a) View the situation from within frame M. Call the time for the light to travel from the emitter to the mirror in this frame t'_1. How far does the mirror move in this time? What distance does the light pulse travel in terms of t'_1 , v, and d'? This distance must equal ct'_1. You should therefore find that

$$t'_1 = \frac{d'}{c-v}$$

ANSWER: The mirror travels a distance $x = vt'_1$. The light pulse travels a distance $d_1 = d' + vt'_1$. Hence $ct'_1 = d' + vt'_1$, so $t'_1 = d'/(c-v)$.

(b) Using the same logic, find the time t'_2 for the light pulse to return from the mirror to the detector in frame M in terms of d', c, and v. Hence find the clock period $T' = t'_1 + t'_2$ in this frame.
ANSWER: For the way back: the light travels a distance $d_2 = d' - vt'_2$. This gives $ct'_2 = d' - vt'_2$ so $t'_2 = d'/(c+v)$. Hence

$$T' = \frac{d'}{c-v} + \frac{d'}{c+v} = \frac{d'(c+v) + d'(c-v)}{c^2 - v^2} = \frac{2d'c}{c^2 - v^2}$$

(c) Which reference frame measures proper time for the light clock?
ANSWER: The clock measures the time between two events; the emission of the light pulse, and the return of the light pulse to the detector. (Note, it does not measure the time the light pulse hits the mirror.) In frame C these two events occur at the same place (i.e. at the position of the emitter/detector). Therefore the clock is at rest in frame C, and so this frame measures proper time for these two events.

(d) We know that the periods in the two frames are related by the time dilation formula $T' = \gamma T$. Rewrite this equation in terms of d', d, v, and c. You should now be able to retrieve the length contraction formula.
ANSWER: The time dilation formula requires

$$\frac{2d'c}{c^2 - v^2} = \gamma \frac{2d}{c}$$

Cancelling the 2 and rearranging the factors of c gives

$$\frac{d'}{1 - v^2/c^2} = \frac{d'}{1-\beta^2} = \gamma^2 d' = \gamma d$$

and so $d' = d/\gamma$ as required.

(e) Show that

$$t'_1 \neq \gamma t_1 \qquad \text{and} \qquad t'_2 \neq \gamma t_2$$

Why do these not hold?
ANSWER: Using the length contraction formula

$$t'_1 = \frac{d'}{c-v} = \frac{d}{\gamma(c-v)} = \frac{\gamma d}{c} \times \frac{1}{\gamma^2(1-\beta)} = \gamma t_1 \times \frac{1}{\gamma^2(1-\beta)}$$

Similarly

$$t_2' = \gamma t_2 \times \frac{1}{\gamma^2(1+\beta)}$$

and so the inequalities stated in the question are correct. Time dilation relates the proper time, measuring in the rest frame of an object, to the time in an inertial frame where the object is moving. The important factor is that the proper time measures the time for an object at rest, i.e. at the same position as a function of time. However, t_1 and t_2 relate to the times between the two ends of the light paths. The two ends are not at the same position, even in the overall light clock rest frame, which means the time dilation formula does not apply.

Multiple choice questions for coursework

1. True or false: "In principle, it is possible for an observer following a pulse of light at a constant high speed to observe the light pulse to be almost stationary."

   (a) True
   (b) False

   ANSWER: (b) False.
   The observer travelling at high speed will still observe the light pulse travelling at speed c away from him (by postulate 2).(Don't confuse this with an observer in a different inertial frame who can see the first observer and the light pulse travelling at nearly the same speed.) [2 marks]

2. Abi is in a spaceship moving at high speed relative to Ben, who is standing on an asteroid (a rock floating in space). She flies past him so that at $t = 0$, she is momentarily adjacent to Ben. At the instant that Abi's spaceship passes Ben, she sends two light pulses to him from her spaceship. If the light pulses are emitted a nanosecond (10^{-9} seconds) apart according to Abi's clock, what will be the time interval between the pulses according to Ben?

   (a) Greater than one nanosecond
   (b) Equal to one nanosecond
   (c) Less than one nanosecond

   ANSWER: (a) Greater than one nanosecond.
   Abi records proper time in this situation (she emits the two light pulses in the same place in her frame). Ben will therefore observe her clock to run slow, and will receive the pulses with a time interval which is $>$ one nanosecond. [2 marks]

3. Also while Abi's spaceship passes Ben, Ben sends two light pulses to Abi. If Ben sends the light pulses a nanosecond apart according to his clock, what will be the time interval between the pulses according to Abi?

   (a) Greater than one nanosecond
   (b) Equal to one nanosecond
   (c) Less than one nanosecond

   ANSWER: (a) Greater than one nanosecond.
   This time the situation is opposite: Ben records proper time as he emits the two light pulses in the same place in his frame. Abi receives the pulses with a time interval which is $>$ one nanosecond. [3 marks]

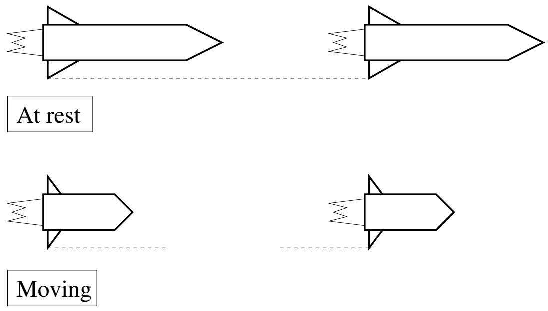
4. Two identical rockets are floating in space one behind the other, at rest relative to an observer. The observer instructs both rockets to fire their engines at exactly the same time and, as they are identical, they then have the same acceleration. After a pre-programmed time, both engines shut off and the rockets drift at some constant speed relative to the observer. Since their accelerations are identical, they have the same distance between them at all times as measured by the observer, including after the engines are turned off.

   As shown in the diagram, a thin piece of thread was tied between the tail fin of the first and second rocket before the engines started, such that it was taut. After the acceleration, when the rockets are drifting, what will the state of the thread be?

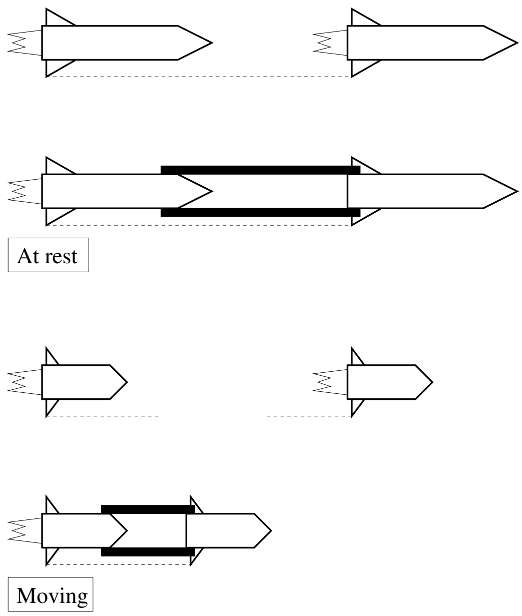
   (a) Still taut.
   (b) Snapped.
   (c) Slack.

ANSWER: (b) Snapped.
Because it was originally taut, the length of the thread in its rest frame is the same as the distance between the rockets. The rockets have the same distance apart when drifting in the observer frame, but the thread rest frame is no longer the observer frame. If it was floating freely, the observer would see the thread Lorentz contracted by $1/\gamma$. As it is tied at both ends, this means it must snap as this is shorter than the distance between the rockets. The rockets themselves also Lorentz contract so the start and end views are as shown below.



This seems to contradict what you might expect from considering the two rockets and thread as one system, as it seems that it would simply all Lorentz contract in the moving frame, with the thread still taut. However, this is not a passive transformation. Because the distance between the rockets in the observer frame (not their rest frame) is forced to be constant by the rocket accelerations, then Lorentz contraction requires that the distance between them in the rocket rest frame is larger, i.e. so that this length still contracts to the constant distance in the observer frame. The pilots on each rocket would see the other rocket getting further and further away while the acceleration was happening.

It might help to contrast this with what would happen if the rockets were bolted together rigidly and only the rear rocket engine was used (but with increased power so as to give the same acceleration for the whole structure). The whole double rocket system would Lorentz contract so the distance between e.g. the two engines would *not* remain the same any more. The two cases are compared below.



[3 marks]