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Relativity - Solutions to Problem Sheet 4

Topics covered: particle collisions, the Relativistic Doppler effect

Questions to try in your own time

1. A particle of mass m, whose total energy is twice its rest energy, collides with an identical particle at rest. If they react to form a new particle, what would be its mass M? Show its velocity is $c/\sqrt{3}$.
ANSWER: The total energy considering the initial system is $E_T = 2mc^2 + mc^2 = 3mc^2$, including both particles, and the total momentum is $p_T c = \sqrt{(2mc^2)^2 - (mc^2)^2} = \sqrt{3}mc^2$, entirely from the first particle. Since energy and momentum are conserved, these must be equal to the energy and momentum of the final particle, so it has a four-momentum $(E_T, \vec{p}_T)$. Hence, its mass must be given by
$$M^2c^4 = E_T^2 - p_T^2c^2 = 9m^2c^4 - 3m^2c^4 = 6m^2c^4 \qquad \text{so} \qquad M = \sqrt{6}m$$
The final particle speed can be found using the standard formula $u = pc^2/E$, so
$$u = \frac{p_T c^2}{E_T} = \frac{\sqrt{3}mc^3}{3mc^2} = \frac{c}{\sqrt{3}}$$

2. Two protons ($m_p = 1.67 \times 10^{-27}$ kg) are initially moving with equal speeds in opposite directions. The protons continue to exist after a collision that also produces an η particle ($m_\eta = 9.75 \times 10^{-28}$ kg).
   (a) If the two protons and the η are all at rest after the collision, find the initial speed of the protons, expressed as a fraction of the speed of light.
   ANSWER: The initial total energy of the two protons is $2\gamma_u m_p c^2$. Since all three particles are at rest after the collision, the final total energy is $2m_pc^2 + m_\eta c^2$. Conservation of energy means we can equate these two:
   $$2m_pc^2 + m_\eta c^2 = 2\gamma_u m_p c^2 \quad \text{so} \quad \gamma_u = 1 + \frac{m_\eta}{2m_p} = 1.29 \quad \text{and} \quad \beta_u = \sqrt{1 - \frac{1}{\gamma_u^2}} = 0.633$$
   Hence the proton speed is $0.633c$.
   (b) What is the kinetic energy of each proton before the collision? Express your answer in MeV.
   ANSWER: The kinetic energy of each proton is $K = (\gamma_u - 1)m_pc^2 = 4.39 \times 10^{-11}\,\text{J} = 274\,\text{MeV}$.
   (c) What is the rest energy of the η, expressed in MeV?
   ANSWER: Rest energy is $m_\eta c^2 = 8.78 \times 10^{-11}\,\text{J} = 548$ MeV.
   (d) Discuss the relationship between the answers to parts (b) and (c).
   ANSWER: The protons lose all the kinetic energy in the collision; this is $2 \times 274 = 548\,\text{MeV}$, which is equal to the energy needed to produce the new particle.

3. Protons and their antimatter equivalent, called antiprotons $\overline{p}$, have identical masses. A proton and antiproton can annihilate each other and one possible outcome is the creation of a positive and negative pion pair, π^+ and π^-
$$p + \overline{p} \quad \rightarrow \quad \pi^+ + \pi^-$$
The rest mass energy of each pion is 140 MeV. Assume the proton and antiproton are travelling with velocities of equal magnitude and opposite direction.
   (a) Does this reaction have a proton/antiproton energy threshold?
   ANSWER: The proton (and hence also antiproton) mass is given in the previous question and so $m_pc^2 = 1.50 \times 10^{-10}\,\text{J} = 939\,\text{MeV}$. This is greater than the pion mass and so there will always be sufficient energy for the reaction, even with if the proton and antiproton have no kinetic energy.
   (b) What is the energy of each pion in terms of the energy of the proton (or antiproton)?
   ANSWER: Since both the initial particles have the same mass as each other, and the final particles also have the same mass as each other, then by symmetry the total energy must be equally divided in both cases. Hence $E_p = E_T/2 = E_\pi$.

(c) If the proton and antiproton are travelling very slowly before the reaction such that their kinetic energy is negligible, what is the kinetic energy of each pion?
ANSWER: In this case $E_p = E_\pi = 939\,\mathrm{MeV}$ so each pion has a kinetic energy $K_\pi = E_\pi - m_\pi c^2 = 799\,\mathrm{MeV}$.

4. Use the quantum Planck-Einstein and de Broglie relations to write the four-momentum for a photon moving in the x direction in terms of its frequency f only. Subsequently, apply a Lorentz transformation to transform to an inertial frame with a speed $v = \beta c$, firstly along the $-x$ axis and secondly along the $+x$ axis. Hence retrieve the Doppler formulæ for both forward (blue) and backward (red) shifts $f_f = f\sqrt{(1+\beta)/(1-\beta)}$ and $f_b = f\sqrt{(1-\beta)/(1+\beta)}$.
ANSWER: The four-momentum for a photon (taking p along the x-axis) is

$$(E, pc, 0, 0) = (hf, \frac{hc}{\lambda}, 0, 0) = (hf, hf, 0, 0)$$

Transforming to a frame moving with $-\beta$ gives

$$E' = hf' = \gamma(E + \beta pc) = \gamma(hf + \beta hf) = hf\gamma(1+\beta)$$

Therefore

$$f_f = f\frac{1}{\sqrt{1-\beta^2}}(1+\beta) = f\sqrt{\frac{(1+\beta)^2}{(1+\beta)(1-\beta)}} = f\sqrt{\frac{1+\beta}{1-\beta}}$$

Clearly, the opposite boost simply changes the sign of β so

$$f_b = f\sqrt{\frac{1-\beta}{1+\beta}}$$

5. In a spiral galaxy individual stars rotate in a flat plane about the galactic centre.

(a) A spiral galaxy with a diameter of 100,000 light-years is observed edge-on from Earth. At either edge of the galaxy, light from a hydrogen emission line is recorded at a wavelength of 536 nm and 803 nm. This emission line has $\lambda = 656\,\mathrm{nm}$ in a lab on Earth. What is the rotational period of this galaxy?
ANSWER: It is easiest to invert Doppler formula in terms of wavelength

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}} \qquad \text{so} \qquad \beta = \frac{1-(\lambda'/\lambda)^2}{1+(\lambda'/\lambda)^2}$$

Putting in the given values gives $v_1 = 0.2c$ and $v_2 = -0.2c$ which means the outer edge of the galaxy is rotating with a speed $0.2c$. The circumference is $\pi \times 10^5$ light-years and hence the rotational period is $\pi \times 10^5/0.2 = 1.58 \times 10^6$ years.

(b) The same hydrogen line in a different spiral galaxy, also observed edge-on, has measured wavelengths of 558 nm and 726 nm at each edge of the galaxy. Calculate the velocities these Doppler shifts imply. Without doing any detailed calculations, comment on the possible motion of this galaxy.
ANSWER: These wavelengths give $v_1 = 0.16c$ and $v_2 = -0.10c$. The galaxy is both rotating and moving towards Earth.

6. Spectators are watching a race between two relatistvistic race cars. They see car A speed towards them at $\beta = 2/3$, whereas car B has gone passed and is speeding away at $\beta = 1/2$. (You can assume the positions of both cars and the spectators lie on a straight line, with the direction of motion of the cars along the line). Both cars are painted yellow with a peak wavelength of 580 nm.

(a) What peak wavelength do the spectators see for each car?
ANSWER: Using the relativistic Doppler shift formula, with the forward case for car A and the backward case for car B

$$\lambda_A = \lambda\sqrt{\frac{1-\beta}{1+\beta}}\lambda_B = \lambda\sqrt{\frac{1+\beta}{1-\beta}}$$

For car A this gives $580\sqrt{(1/3)/(5/3)} = 259\,\mathrm{nm}$ (blue-shifted), while for car B this gives $580\sqrt{(3/2)/(1/2)} = 1005\,\mathrm{nm}$ (red-shifted).

(b) What peak wavelength does car B look like to car A? What about car A to car B?
ANSWER: To get the right velocity for car B in car A's rest frame, use the velocity addition formula to obtain

$$\beta' = \frac{(1/2) - (2/3)}{1 - (1/2)(2/3)} = -\frac{1}{4}$$

This is negative because car A is going faster than car B in the spectator frame. Therefore car B is approaching car A in car A's rest frame with speed 1/4. Hence we use the forward Doppler formula with 1/4 which gives $580\sqrt{(3/4)/(5/4)} = 449\,\text{nm}$ (blue-shifted). Without any calculation we know that car A's peak wavelength will also be observed at 449 nm by car B, as the relativistic Doppler effect is entirely symmetrical: we cannot distinguish between inertial frames.

(c) Now assume cars A and B are travelling much slower, at $v_A = 2w/3$ and $v_B = w/2$, where w is the speed of sound, and the cars' engines produce a sound at 500 Hz. Using the classical Doppler equation, what sound frequency does car B record from car A's engine? What about car A from car B? Explain the differences between this scenario and the previous one.
ANSWER: We now need to use the classical Doppler effect, not only because the speeds are much lower than the speed of light, but because we also need to account for the fact that sound moves through a medium. This means the Doppler effect is not symmetrical for a moving source and moving detector. The classical Doppler effect for a source moving towards a forward detector is:

$$f_{fS} = \frac{f}{1 - (v_S/w)}$$

and for a moving detector we have:

$$f_{fD} = f\ [1 + (v_D/w)]$$

For a moving source AND moving detector we need to combine the two formulæ

$$f_{fDS} = f\ \frac{1 + (v_D/w)}{1 - (v_S/w)}$$

For car A recording the sound of car B, $v_S = -w/2$, and $v_D = 2w/3$. This gives us $f_{fDS} = 500 \times (1 + 2/3)/(1 + 1/2) = 556\,\text{Hz}$. For car B recording the sound of car A, $v_S = 2w/3$, and $v_D = -w/2$. The result is $f_{fDS} = 500 \times (1 - 1/2)/(1 - 2/3) = 750\,\text{Hz}$.

The differences, as already mentioned above, are (i) using the classical Doppler effect we are not taking time dilation into account, so there is no γ factor in the formula, and (ii) the Doppler effect for sound is not symmetrical for observer or source motion. We could use this effect distinguish between reference frames by comparing them to the frame where the sound medium (air) is stationary.

7. **Tutorial problem: fixed target vs colliding beam experiments**

The first particle physics experiments involved accelerating a beam of protons to high energy and then colliding them with a target at rest. A target such as liquid hydrogen would result in proton-proton reactions.

This is technically simple, but an inefficient way to make new particles because some of the initial energy goes into the kinetic energy of the collision products, rather than into their mass. It is better to collide two beams of protons head-on. This question compares the energy E^* needed by a proton hitting another at rest to give the same centre-of-mass (CM) energy E_{CM} as two colliding protons of energy E each.

(a) Classically, the only relevant energy is kinetic, K. Using classical calculations, show that to get the same CM energy, the kinetic energy K^* required by a proton hitting another at rest must be $4K$, where K is the energy of each proton when colliding head-on. What fraction of K^* appears as the CM energy?
ANSWER: Since the total momentum of the two colliding protons is zero, then all the energy is the CM energy, so it is $2K$. If the speed of each proton is u, then $K = m_p u^2/2$. By transforming by speed u, then one of the protons will be at rest and the other will have $u' = 2u$. This is therefore the case with a proton at rest and the other moving. In this frame, then $K^* = m_p u'^2/2 = 4m_p u^2/2 = 4K$. The CM energy is $E_{\mathrm{CM}} = 2K$ which is therefore half of K^*.

(b) Still working classically, consider both protons as one "system" for the case when one is at rest. What is the total mass and momentum of this system? Treating this as one classical object, calculate its kinetic energy. What fraction of K^* appears as this overall kinetic energy?
ANSWER: The total mass of the two-proton system is $2m_p$ and its total momentum is $2m_p u$. Hence, the kinetic energy of the system is $K_S = 2m_p u^2/2 = m_p u^2$. This is also half of K^*, so we find half the incoming proton energy appears in the CM but the other half is needed to keep the overall system moving, since it has non-zero momentum.

(c) The additional beam energy needed classically would not be so terrible. However, the actual additional energy needed relativistically can be enormous. Starting from the CM frame, use the Lorentz transformations for energy and momentum to show that in the frame where one of the protons is at rest, the other has an energy

$$E^* = \frac{2E^2}{m_p c^2} - m_p c^2$$

ANSWER: In the CM frame, both protons have equal and opposite momentum. Each has energy E and let p be their momentum. We know $\beta_u = u/c = pc/E$ and $\gamma_u = E/m_p c^2$. We need to transform to a frame where one proton is at rest, which means a Lorentz transformation using $\beta = \beta_u$. The energy of the other proton E^* is

$$E^* = \gamma_u(E+\beta_u pc) = \frac{E}{mc^2}\left(E + \frac{p^2c^2}{E}\right) = \frac{E^2}{m_p c^2} + \frac{p^2c^2}{m_p c^2} = \frac{E^2}{m_p c^2} + \frac{E^2 - m_p^2 c^4}{m_p c^2} = \frac{2E^2}{m_p c^2} - m_p c^2$$

(d) Express E and E^* in terms of relativistic kinetic energy $K = E - mc^2$. Hence show that the above expression approximates to the classical result when the kinetic energy is small compared with the rest mass energy.
ANSWER: With $E = K + m_p c^2$ and $E^* = K^* + m_p c^2$, then the above becomes

$$K^* + m_p c^2 = \frac{2(K + m_p c^2)^2}{m_p c^2} - m_p c^2 = \frac{2K^2 + 4K m_p c^2 + 2m_p^2 c^4}{m_p c^2} - m_p c^2 = 4K + \frac{2K^2}{m_p c^2} + m_p c^2$$

Hence

$$K^* = 4K\left(1 + \frac{K}{2m_p c^2}\right)$$

This gives $K^* \approx 4K$ for $K \ll m_p c^2$, which is the classical limit.

(e) The LHC currently collides protons with $E = 6.5\,\mathrm{TeV}$ each. Take $m_p = 1\,\mathrm{GeV}$. What is E^*? What multiple of E would this amount to? This explains why modern accelerators always need to face the huge technical challenge of colliding two beams.
ANSWER: Putting in the numbers in TeV gives

$$E^* = \frac{2 \times 6.5^2}{0.001} - 0.001 = 8.4 \times 10^4\,\mathrm{TeV}$$

Hence $E^*/E = 1.3 \times 10^4$ is the factor of extra energy needed using one proton at rest. This is not (currently) technically possible, as $6.5\,\mathrm{TeV}$ is already near the limit of our achievable beam energy. The extra technical complications of a colliding beam experiment are well worthwhile.