Lecture 7 Impedance

Topics:
Reactance of capacitor and inductor, complex impedance, equivalent impedances, AC analysis

Our aim here is to produce a generalised form of Ohm's law which includes also capacitors and inductors. Returning to the discussion of linearity in section 6.4, we have

$$v_R = i_R R$$
$$v_L = L\frac{di_L}{dt}$$
$$i_C = C\frac{dv_C}{dt}$$

7.1 Reactance

For the resistor, the *amplitudes* of the voltage and current are linearly related, and of course the current is always in-phase with the voltage. The constant of proportionality in this linear relation is the **resistance**, with units of ohms. The equivalent quantity for capacitors and inductors is called the **reactance**, also with units of ohms.

Reactance of the Capacitor

For

$$v_C = A\cos(\omega t)$$

the current through the capacitor is

$$i_C = C\frac{dv_C}{dt} = -A\omega C\sin(\omega t) = A\omega C\cos\left(\omega t + \frac{\pi}{2}\right) \tag{42}$$

Note that we have chosen $\phi = 0$ for simplicity. The reactance of the capacitor is the ratio of the voltage *amplitude* to the current *amplitude*

$$X_C = \frac{|v_c|}{|i_C|} = \frac{1}{\omega C} \tag{43}$$

Note that the capacitor reactance is **frequency-dependent**.

Reactance of the Inductor

Similarly for the inductor

$$i_L = A\cos(\omega t)$$

$$v_L = L\frac{di_L}{dt} = -A\omega L\sin(\omega t) = A\omega L\cos\left(\omega t + \frac{\pi}{2}\right) \tag{44}$$

$$X_L = \frac{|v_L|}{|i_L|} = \omega L \tag{45}$$

The inductor reactance is also frequency-dependent.

7.2 Complex Impedance

Equations 43 and 45 would be sufficient if we just want to know about the amplitudes of the signals, but we can use the complex representation to additionally encode the phase changes involved.

Complex Impedance of the Inductor

Equation 44 tells us that the voltage across the inductor **leads** the current by $\pi/2$ radians so we can write the **complex impedance** of the inductor as

$$\tilde{Z}_L = jX_L = j\omega L \tag{46}$$

Complex Impedance of the Capacitor

Equation 42 tells us that the voltage across the capacitor **lags** the current by $\pi/2$ radians so we can write the **complex impedance** of the capacitor as

$$\tilde{Z}_C = -jX_C = \frac{-j}{\omega C} \tag{47}$$

Complex Impedance of the Resistor

For completeness, the voltage across the resistor is always in phase with the current (Ohm's law), so the **complex impedance** of the resistor is

$$\tilde{Z}_R = R \tag{48}$$

Note that the resistor impedance is entirely real, representing the in-phase nature of the voltage and current signals, while the capacitor and inductor impedances are entirely imaginary, encoding the $\pi/2$ phase shifts between voltage and current.

Since complex impedance is always a ratio of voltage to current, it has units of ohms (Ω).

7.3 Equivalent Impedance

Complex impedances combine like resistances.

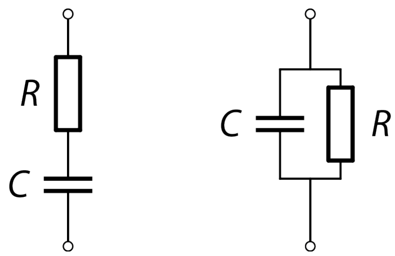

Figure 27: Equivalent impedances: series (left) and parallel (right)

Impedances in Series

The equivalent impedance of series impedances is

$$\tilde{Z}_{eq} = \tilde{Z}_1 + \tilde{Z}_2 + \ldots \tag{49}$$

For example, in figure 27 (left), the combined total impedance $\tilde{Z}$ is given by

$$\tilde{Z} = \tilde{Z}_R + \tilde{Z}_C = R - \frac{j}{\omega_C}$$

Impedances in Parallel

The equivalent impedance of parallel impedances (e.g. figure 27, right) is

$$\frac{1}{\tilde{Z}_{eq}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \ldots \tag{50}$$

Consequently, combining resistances with **reactive** components results in an equivalent impedance with *both real and imaginary* parts.

7.4 AC Analysis

As long as we follow all the rules for manipulation of complex quantities, we can use complex impedances just like resistances with Ohm's law. Using the phasor representation for voltage $\tilde{v}$ and current $\tilde{i}$

$$\tilde{Z} = \frac{\tilde{v}}{\tilde{i}}$$

So we can find an unknown voltage from a current by

$$\tilde{v} = \tilde{i}\tilde{Z}$$

Finding the Phase

We can determine the phase shift of the voltage with respect to the current by plotting the result on a phasor diagram. We can also note that the phase shift will be the argument of $\tilde{Z}$. To understand this, it again helps to use the complex exponential form. For example, if $\tilde{Z}$ represents an inductor then

$$\tilde{Z}_L = j\omega L = e^{j\pi/2}\omega L$$

Consequently, multiplying a current phasor $\tilde{i}$ by an inductor impedance $\tilde{Z}_L$ results in a voltage phasor $\tilde{v}$ which must lead the current by $\pi/2$ radians.

Magnitude of the Impedance

Furthermore, the **amplitudes** of the voltages and currents are related by the **magnitude** of the impedance.

$$|\tilde{Z}| = \frac{|\tilde{v}|}{|\tilde{i}|}$$

Since all our signals are sinusoidal, the RMS amplitude is related to the signal amplitude by a factor $1/\sqrt{2}$ and we can use

$$V_{RMS} = |\tilde{Z}| I_{RMS}$$

etc.

Circuit Rules

The circuit rules of section 2 can be extended to AC signals. Kirchhoff's voltage law always applies in that the *instantaneous* sum of voltages around any loop must always be zero. On a phasor diagram, this means that the vector sum of voltages is zero, i.e. both the real and imaginary voltage components sum to zero. We can also plot currents as phasors and Kirchhoff's current law also applies; the vector sum of currents into a junction is zero. We can also use the complex equivalents of the voltage and current divider rules. The use of these principles is best understood by some examples as we will see in section 8. Finally, the principle of superposition still applies, and we can make Thévenin equivalent circuits with complex impedances.

Using complex impedances requires care and a bit of practice, but so long as we follow all the rules for complex numbers and the relationships described above, the results fall-out remarkably easily. The expressions for complex impedance (equations 46 to 48) are usually easiest to remember in the cartesian form, but it's often easiest to convert combined impedances into exponential form in order to simplify the maths.

LTSpice Exercises

> Download the LTSpice schematics drivenC.asc and drivenL.asc from Blackboard

The first schematic shows a capacitor $C = 1000\,\mu F$ driven by a source of the form $v = 10\cos 100t$.

1. What's the capacitor's impedance at this frequency?
2. Given the magnitude of the impedance and the amplitude of the applied voltage, predict the amplitude of the resulting current.
3. Run the simulation to verify this. Plot the current through the capacitor.
4. Write the applied voltage in phasor form and, using $\tilde{i} = \tilde{v}/\tilde{Z}_C$, show that this method gives the amplitude and phase of the current consistent with the simulation.
5. Does the current lead or lag the voltage, and by how many radians?

The second schematic shows an inductor $L = 1\,\text{mH}$ driven by a *current* source of the form $i = 10\cos 100t$.

1. What's the inductor's impedance at this frequency?
2. Predict the amplitude of the resulting voltage.
3. Run the simulation to verify this. Plot the voltage across the inductor.
4. Write the applied current in phasor form and, using $\tilde{v} = \tilde{i}\tilde{Z}_L$, show that this method gives the amplitude and phase of the voltage consistent with the simulation.
5. Does the current lead or lag the voltage, and by how many radians?

Lecture 8 Filters

Topics:
Impedance divider, gain, Bode plot, decibels, roll-off, low-pass and high-pass RC filters, low-pass and high-pass RL filters, filter applications.

We will now re-visit the series RC circuit in order to understand its behaviour in response to AC signals. The frequency-dependence of the capacitor's impedance implies that we can use the RC circuit as a **filter**, that is to say, a circuit which will pass through some frequencies and block others. The RC filter of figure 28 consists of a resistor and capacitor in series; the input signal V_{in} is applied to the series combination while the output V_{out} is the voltage across the capacitor alone. Since the capacitor's impedance is inversely proportional to frequency, we can make some predictions about the behaviour:

1. For low frequencies, $\omega \to 0$, the capacitor's impedance $|Z_C| \to \infty$ so no current will be drawn from the supply, there will be no voltage drop across R, and hence $v_{out} = v_{in}$.
2. For high frequencies, $\omega \to \infty$, the capacitor's impedance $|Z_C| \to 0$ so there will be a voltage drop across R equal to v_{in}, and hence $v_{out} = 0$.

Consequently, figure 28 is described as a **low-pass filter** since it passes low-frequencies and blocks high-frequencies.

8.1 Impedance Divider

R and C form an **impedance-divider** much as two series resistors form a resistive-divider. The combined impedance in series is

$$\tilde{Z}_R + \tilde{Z}_C = R - \frac{j}{\omega C}$$

The output voltage is

$$\tilde{v}_{out} = \tilde{v}_{in} \times \frac{\tilde{Z}_C}{\tilde{Z}_R + \tilde{Z}_C} = \tilde{v}_{in} \times \frac{-j/\omega C}{R - j/\omega C}$$

8.2 Gain

The filter is usually characterised by the **gain**, which is the ratio of output to input voltages.

$$\tilde{g} = \frac{\tilde{v}_{out}}{\tilde{v}_{in}} = \frac{1}{1 + j\omega RC} \tag{51}$$

It is useful to know the magnitude and argument of the complex gain.

$$g = |\tilde{g}| = \frac{1}{\sqrt{1 + (\omega CR)^2}} \tag{52}$$

$$\tan \phi = -\omega RC$$

$$\phi = -\tan^{-1} \omega RC \tag{53}$$

Equation 52 validates the two earlier predictions about the behaviour for low and high frequencies.

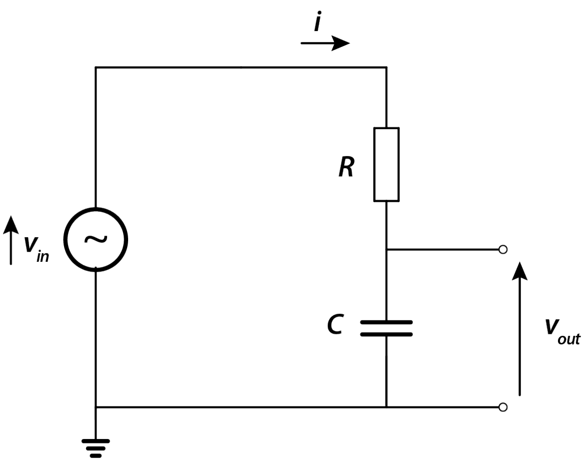

Figure 28: Low-Pass RC Filter

8.3 Bode Plot

To visualise g it is convenient to plot the **magnitude**, versus frequency, on a log-log plot. The argument of the gain, or **phase**, is plotted on a log-linear plot. Together, these two plots are commonly known as the **Bode Plot**. The general form is shown in figure 29, which uses $RC = 1\,\mathrm{s}$ for convenience.

Decibels

The logarithm of the gain magnitude is, by convention, expressed in dB (decibels).

$$g_{\mathrm{dB}} = 20 \log_{10} g \tag{54}$$

The decibel is a dimensionless measure of **relative power** defined as

$$P_{\mathrm{dB}} = 10 \log_{10} \frac{P_2}{P_1}$$

P_{dB} expresses the power P_2 relative to P_1 so if the ratio is 100 then $P_{\mathrm{dB}} = 20\,\mathrm{dB}$ and if the ratio is $1/2$ then $P_{\mathrm{dB}} = -3.01\,\mathrm{dB}$. Whilst the decibel always expresses relative power, we can consider the power in a signal to be proportional to the square of the amplitude ($P = V^2/R$) so we can express the ratio of two *amplitudes* as

$$P_{\mathrm{dB}} = 20 \log_{10} \frac{V_2}{V_1}$$

Since g is a ratio of amplitudes, equation 54 expresses the relative power in the output of the filter to the input of the filter.

Cut-off Frequency

All low-pass RC filters will have the same general shape of figure 29. For low frequencies, g is always 1 (0 dB) but the frequency at which the filter starts to attenuate can be adjusted by changing the

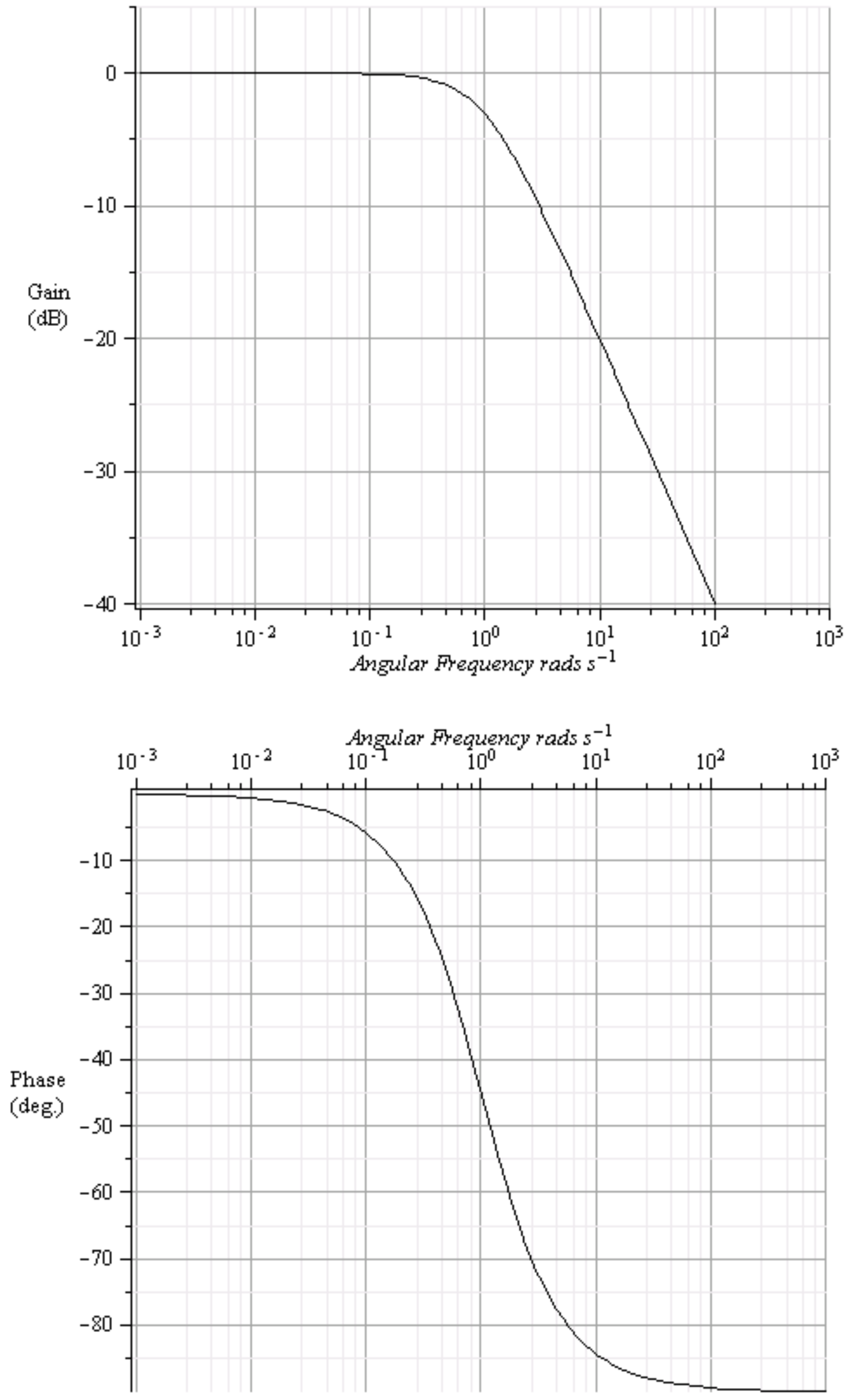

Figure 29: Magnitude and Gain of the Low Pass Filter (normalised to $RC = 1$).

values R and C. Since there is no sharp transition from the **pass-band** to the **roll-off**, by convention we choose the frequency where the output power is half the input which can be found by solving

$$\frac{1}{2} = \frac{1}{1 + (\omega_c CR)^2}$$

which gives the **cut-off angular frequency**

$$\omega_c = \frac{1}{RC} = \frac{1}{\tau}$$

where τ is the **time-constant** of the RC circuit.

At the frequency ω_c, the gain is $g = 1/\sqrt{2}$ so $g_{dB} = -3.01\,\text{dB}$. Consequently, the top curve in figure 29 will always have a value of -3 dB at this frequency. Note also that the phase will always be $-45°$.

Roll-Off

An ideal low-pass filter would pass all frequencies below the cut-off and block all frequencies above. This is impossible to achieve in real-world filters. For the RC low-pass filter, it can be seen that for $\omega \gg \omega_c$,

$$g \approx \frac{1}{\omega RC}$$

which results in a gradient of -20 dB/decade, where a decade is a factor of 10 in frequency. For example, the gain falls by -20 dB between the frequencies 10 and 100 rad/s.

Phase

Conventionally, the phase of the Bode plot is expressed linearly in degrees. The phase of the low-pass filter expresses the phase change between the output versus the input and the negative sign indicates that the phase of the output lags the input. We can see this by re-arranging equation 51:

$$\tilde{v}_{out} = \tilde{v}_{in}\tilde{g}$$

Taking an arbitrary input signal $\tilde{v}_{in} = V_0$ (i.e. an input signal with amplitude V_0, phase angle zero, and an assumed frequency ω) then the output will be

$$\tilde{v}_{out} = V_0 g e^{\phi}$$

where g and ϕ are functions of R, C and ω and defined by equations 52 and 53 respectively. For the low-pass RC filter of figure 28, ϕ will be negative. The amplitude of the output sinusoid is $v_{out} = V_0 g$ and $v_{out_{RMS}} = V_0 g/\sqrt{2}$.

8.4 Low-Pass and High-Pass Filters

The circuit of figure 28 is a *low-pass filter*, as shown by equation 52. We can turn this into a **high-pass filter** by swapping the resistor and the capacitor (this means we take the output voltage across the resistor). Performing the same circuit analysis gives the gain as

$$g = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

$$\phi = \tan^{-1} \omega RC$$

The cut-off frequency is again

$$\omega_c = \frac{1}{RC}$$

For low frequencies, $g \propto \omega$ (the gradient is $+20\,\text{dB/decade}$) while for $\omega \gg \omega_c$, $g \to 1$.

8.5 RL Filters

Since the impedance of the inductor is also frequency-dependent, we can build RL filters with low and high-pass characteristics. In the circuit of figure 28 the capacitor can be replaced with an inductor. $|Z_L| \propto \omega$ so this is a high-pass RL filter. The same principles, starting with the impedance divider, can be used to derive the gain of the RL filter, and again we can make a low-pass RL filter by swapping the resistor and inductor.

8.6 Filter Applications

Filters are often shown drawn as in figure 30 in order to indicate that the input signal V_{in} comes in from some **source** (not shown but assumed to be on the left) and the output signal V_{out} goes on to some **load** (not shown but assumed to be on the right). The purpose of the filter is to modify the frequency content of the signal.

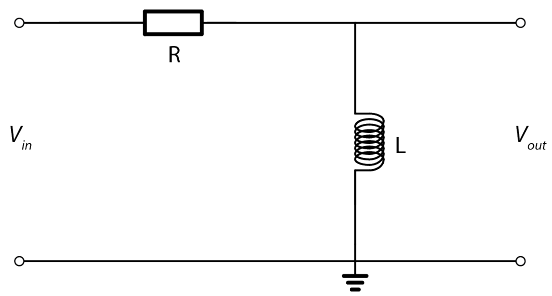

Figure 30: High-pass RL filter.

Audio Example

The source could be a microphone, which is a sensor with characteristics of a voltage source. The microphone might be sensitive to frequencies up to 50 kHz, but the human ear can't hear hear much above 16 kHz or so. Consequently, it makes sense to 'filter-out' all frequencies above 16 kHz before amplifying the signal. The amplifier, typically known as a 'pre-amp' in audio applications, would appear as a load resistance. If we wish to study the setup, the circuit would look like figure 31.
The microphone is represented as a voltage source V_S (note that in practice this may have some small output resistance, typically $R_S = 150\,\Omega$ or so). The pre-amp is represented by a load resistor R_L. A typical pre-amp would have an **input resistance** $R_L = 10\,\text{k}\Omega$. The time-constant $\tau = RC$ would be chosen to give a filter-cut-off frequency about 16 kHz. Typically, we would use a value of R which satisfies $R_S \ll R \ll R_L$ so as to avoid the filter loading the source and the 'load' loading the filter.

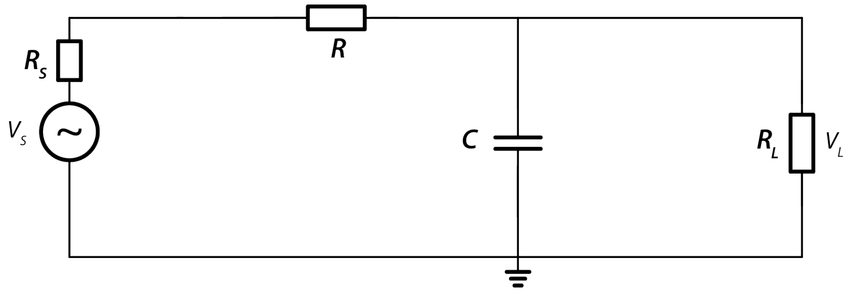

Figure 31: A low-pass filter used in an audio example. The source may be a microphone and the load may be a pre-amplifier.

LTSpice Exercises

Download the LTSpice schematics LowPassFilter.asc and LowPassFilter-Bode.asc

The schematic of LowPassFilter.asc uses components from the set of standard values for resistors and components to build a low-pass filter with a cut-off frequency at approximately 16 kHz.

1. Run the simulation and plot the applied voltage and the output voltage. At 1 kHz (well below the cut-off) the traces should be very similar.
2. Edit the source properties to increase the frequency in factors of two (2 kHz, 4 kHz, etc. . . 32 kHz) and note the approximate amplitude of the output at each frequency.
3. Check the result that, at the cut-off frequency, the output amplitude should be a factor $1/\sqrt{2}$ times the input (equivalent to -3 db).
4. Again at the cut-off frequency, make an estimation of the phase difference between input and output. Which leads? Does this agree with theory?
5. Construct a phasor diagram showing the filter at the cut-off frequency.

To make the simulation more 'realistic' you could repeat the above after placing a 10 kΩ 'load' resistor in parallel with the capacitor, between V_{out} and GND, and check the result that the power in the load resistor reduces with frequency (to about one-half at the cut-off).

LTSpice can automate these frequency scans to create a Bode Plot. Open the schematic LowPassFilterBode.asc.

1. Note the use of the .ac dec. . . spice directive, which tells the simulation to perform an AC analysis.
2. Run the simulation and plot the output voltage. You should get a Bode Plot with the gain magnitude (in dB) and the gain phase (in degrees) in the same plot panel.
3. Click on the V(vout) legend above the plot to bring up cursors and find the frequency at which the gain is -3 dB. Find also the phase at this frequency, and check agreement with theory.
4. Right-clicking on the left-hand vertical scale allows changing the scale between dB, logarithmic and linear. Check that you are able to interpret gain in both linear and decibel representations.

You could also experiment with a high-pass filter, and/or filters built using inductors instead of capacitors.

Lecture 9 Driven LCR Circuit

Topics: Equivalence to the mechanical oscillator, Steady-State AC Analysis, Resonance.

We will now look again at the LCR circuit; here the circuit will be connected to an external EMF source which makes the driven LCR circuit the exact physical analogue of the forced mass-spring-damper mechanical system. The series LCR circuit is shown in figure 32. Kirchhoff gives

$$v = v_L + v_R + v_C$$

Assuming that the circuit is driven by a voltage source with an EMF of the form $v = V_0 \cos \omega t$

$$V_0 \cos \omega t = L\frac{di}{dt} + Ri + v_C$$

Rewriting in terms of charge q

$$V_0 \cos \omega t = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

Using $\gamma = R/L$ and $\omega_0^2 = 1/LC$ gives

$$\frac{V_0}{L} \cos \omega t = \ddot{q} + \gamma\dot{q} + \omega_0^2 q \tag{55}$$

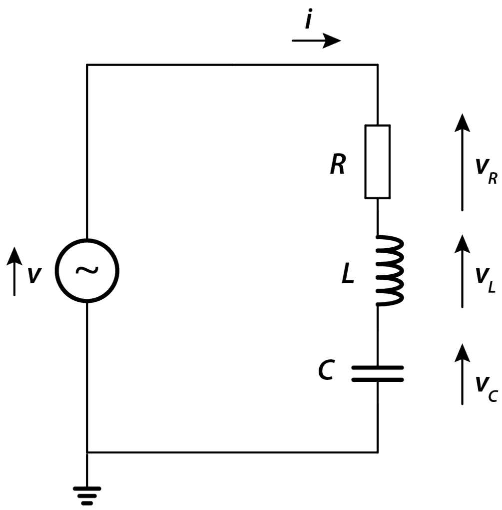

Figure 32: Driven Series LCR Circuit

9.1 Equivalence to the Mechanical Oscillator

Equation 55 has exactly the same form as the differential equation for the displacement x of the mass in the mass-spring-damper system:

$$\frac{F_0}{m} \cos \omega t = \ddot{x} + \gamma\dot{x} + \omega_0^2 x$$

Table 6 summarises the equivalences. The general solution to equation 55 was covered in the Vibrations & Waves course and will not be part of this course but it is useful to remember that the general solution was a superposition of the **transient** (or 'startup') behaviour with the **steady-state**. We will not look at the transient behaviour but will see how we can apply AC circuit analysis to understand the circuit's steady-state behaviour.

Table 6: Mechanical and Electrical Oscillator Equivalences

Mechanical	Electrical
Displacement x	Charge q
Velocity $\dot{x}$	Current $\dot{q}$
Mass m	Inductance L
Spring constant k	Capacitance $1/C$
Damping b	Resistance R
Force F_0	EMF V_0

9.2 Steady-state AC Analysis

The steady-state is reached at a time much after the source has been switched-on, such that any initial transient behaviour has died-down and we are left with a repetitive periodic behaviour. AC circuit theory provides an easy method to characterise the LCR circuit in the steady-state.

The complex impedance of the series LCR circuit is

$$\tilde{Z} = R + j\omega L - \frac{j}{\omega C}$$

The behaviour can be characterised by assuming that the circuit is driven with a signal of the form $v = V_0 \cos \omega t$ which can be represented by a phasor $\tilde{v} = V_0$. The current in the circuit is

$$\tilde{i} = \frac{V_0}{\tilde{Z}}$$

The voltages across the three components can be found from

$$\tilde{v}_R = \tilde{i} R$$

$$\tilde{v}_L = \tilde{i} \tilde{Z}_L$$

$$\tilde{v}_C = \tilde{i} \tilde{Z}_C$$

Since the phase angle of the applied voltage $\tilde{v}$ is defined to be zero, the arguments of the complex current and voltages are their phase angles relative to the applied voltage. It is helpful to plot these on a phasor diagram, and it will be found that the magnitudes and phases will all be frequency-dependent.

Frequency Response

The magnitude and phase of the current are represented in figures 33 and 34. It is instructive to go through the complex algebra to find expressions for $|\tilde{i}|$ and ϕ assuming $\tilde{i} = I_0 e^{j\phi}$. For simplicity, the figures 33 and 34 are calculated for $R = L = C = 1$.

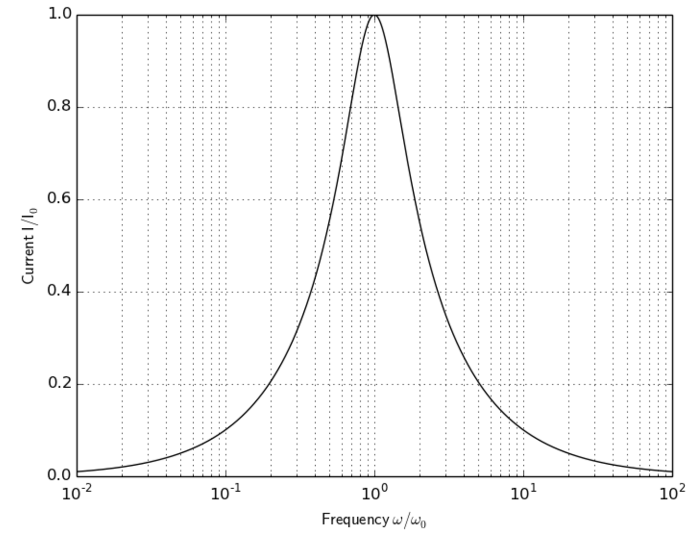

Figure 33: Magnitude of the Current versus Frequency

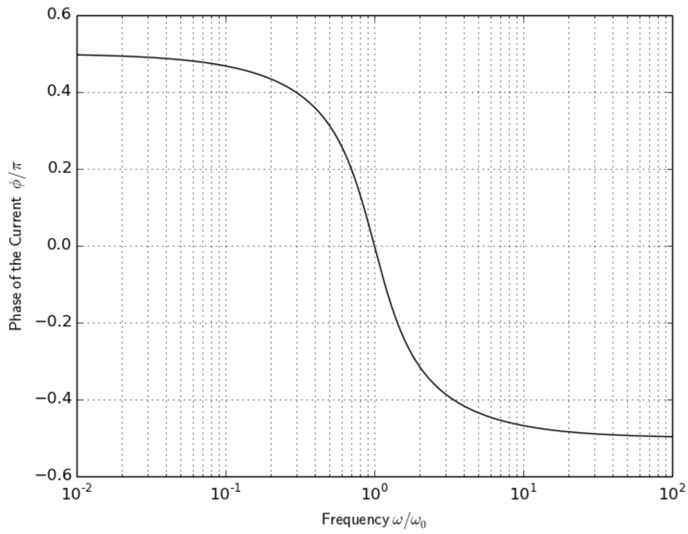

Figure 34: Phase of the Current versus Frequency

LCR Circuit as a Filter

Note that the LCR circuit's frequency-response shows that this type of circuit is neither a high-pass nor a low-pass filter but one which can be used to pass a narrow range of frequencies which we can set by adjusting the values of L, C and R. We could, for example, take the output of the filter as the voltage across the resistor, which will always be in-phase with the current. The LCR circuit is useful for **tuning** applications where a particular frequency needs to be extracted from a signal. Tuning circuits in radios used to use this technique where a variable-capacitor was used to adjust the tuning frequency ω_0.

9.3 Resonance

For $\omega \to 0$ the impedance of the capacitor $|\tilde{Z}_C| \to \infty$ and hence $|\tilde{i}| \to 0$. For $\omega \to \infty$ the impedance of the inductor $|\tilde{Z}_L| \to \infty$. The real (resistive) part of $\tilde{Z}$ is fixed but the imaginary (reactive) part is a function of frequency and it is clear that $|\tilde{Z}|$ will be minimised for the frequency at which $\text{Im}(\tilde{Z})$ is zero:

$$j\omega L = \frac{j}{\omega C}$$

which occurs at the natural frequency

$$\omega = \omega_0 = \sqrt{\frac{1}{LC}}$$

Hence, when the voltage source is oscillating at the natural frequency ω_0

- The impedance is at a minimum.
- The current is at a maximum.
- The impedance is entirely real and equal to the resistance R
- $|\tilde{i}(\omega_0)| = V_0/R$
- The voltage applied across the LCR series circuit will be in phase with the current.
- The voltage across the resistor will be at a maximum.
- The voltage across the inductor and capacitor will be equal in amplitude but opposite in phase.

Power Dissipation

Over a complete cycle, the average power of the inductor and capacitor are zero. Only the resistor dissipates power according to

$$p(t) = v_R(t)i(t) = i(t)^2R$$

We know that the voltage and current in the resistor are always in phase. The average power dissipated is then

$$\langle p(t)\rangle = \frac{I_0^2R}{2}$$

Q-factor

The shape of the curve in figure 33 is characterised by a dimensionless parameter Q known as the **Q-factor** or **quality-factor**.

$$Q = \frac{\omega_0}{\Delta\omega} \tag{56}$$

$\Delta\omega$ is a measure of the width of the peak so it is clear that a high Q-factor is associated with a narrow peak. $\Delta\omega$ is defined as the difference in frequencies between the points where the power dissipated falls to one-half of the maximum. This is illustrated in figure 35 which shows the average power dissipated in an LCR circuit for two different Q-factors. It is left as an exercise to show that, for an lightly-damped LCR circuit (i.e. $\gamma \ll 2\omega_0$)

$$\Delta\omega \approx \gamma = \frac{R}{L}$$

Consequently, reducing the resistance in an LCR circuit increases the 'sharpness' of the resonance peak.

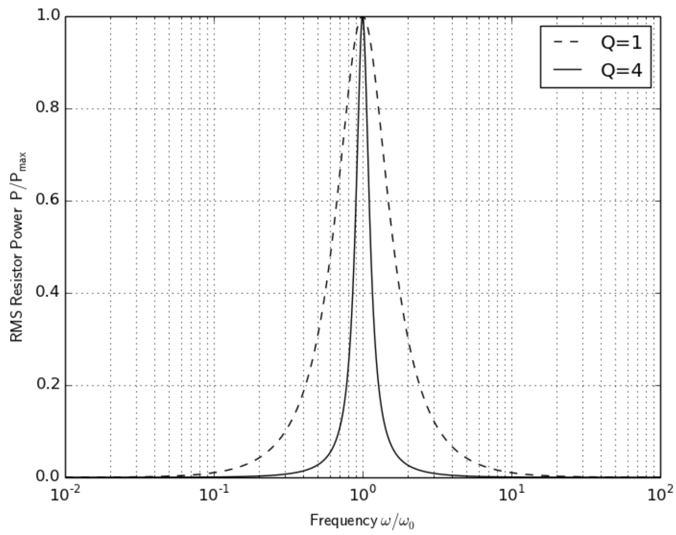

Figure 35: Power Dissipation (normalised to P_{max}) for $Q = 1$ and $Q = 4$

9.4 Worked Example

An LCR circuit with $L = 0.1\,\text{H}$, $C = 0.1\,\text{F}$ and $R = 0.1\,\Omega$ has a natural frequency $\omega_0 = 10\,\text{rad/s}$ and $\gamma = 1$. $Q \approx \omega_0/\gamma = 10$ so the circuit is lightly damped and we can expect a sharp resonance peak. For this example, the circuit is driven at the resonant frequency, i.e. $\omega = \omega_0$, and we'll take the amplitude of the driving voltage to be 1 V for simplicity. As a phasor this is

$$\tilde{v} = 1$$

We can find the impedances of the three components

$$\tilde{Z}_R = 0.1\,\Omega$$

$$\tilde{Z}_L = j\,\Omega$$

$$\tilde{Z}_C = -j\,\Omega$$

Therefore the total series impedance is

$$\tilde{Z} = 0.1\,\Omega$$

We can find the current to be

$$\tilde{i} = \frac{\tilde{v}}{\tilde{Z}} = 10\,\mathrm{A}$$

The current phasor is entirely real and in phase with the voltage. We use the current to find the voltages across the components:

$$\tilde{v}_R = \tilde{i}\tilde{Z}_R = 1\,\mathrm{V}$$
$$\tilde{v}_L = \tilde{i}\tilde{Z}_L = 10j\,\mathrm{V}$$
$$\tilde{v}_C = \tilde{i}\tilde{Z}_C = -10j\,\mathrm{V}$$

We can see that at resonance, the voltage across the resistor is equal to the applied voltage in both magnitude and phase, which we expect as the impedance is entirely real (resistive). The amplitudes of the voltages across the reactive components are much larger, equal, and π radians out of phase with each other. It is helpful to visualise on the phasor diagram (figure 36).

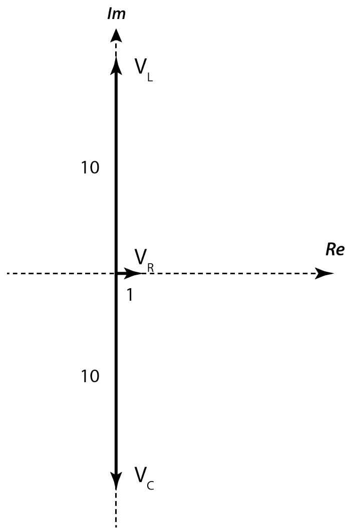

Figure 36: Phasor Diagram for LCR Circuit at Resonance (Q=10)

To get the measurable, physical signal, we can add the time-dependence and take the real part

$$v_R = \mathrm{Re}\left\{1e^{j0}e^{j\omega t}\right\} = \cos(10t)$$
$$v_L = \mathrm{Re}\left\{10e^{j\pi/2}e^{j\omega t}\right\} = 10\cos(10t + \pi/2)$$
$$v_C = \mathrm{Re}\left\{10e^{-j\pi/2}e^{j\omega t}\right\} = 10\cos(10t - \pi/2)$$

These voltages are shown in figure 37

At resonance, the average power is

$$\langle P\rangle = \frac{|\tilde{v}_R|^2}{2R} = 5\,\mathrm{W}$$

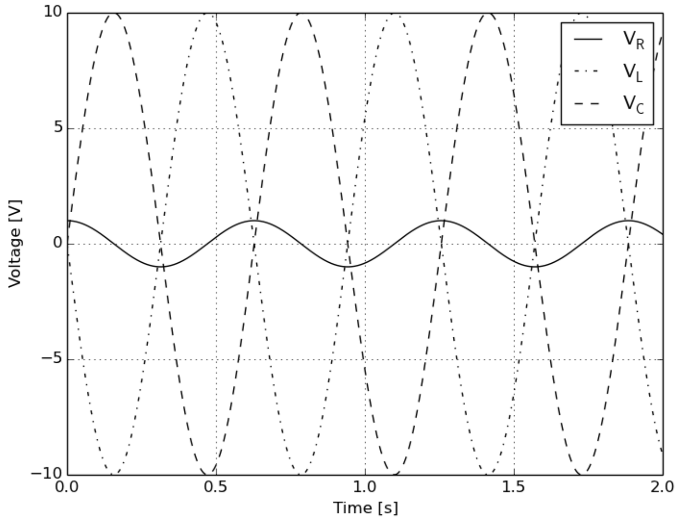

Figure 37: LCR Circuit Voltages at Resonance (applied voltage 1 V)

Theory predicts that the power should drop to approximately half this value at a frequency

$$\omega = \omega_0 \pm \frac{\gamma}{2}$$

To test this, let's choose $\omega = 9.5\,\mathrm{rad/s}$.

$$\tilde{Z}_R = 0.1\,\Omega$$

$$\tilde{Z}_L = 0.95j\,\Omega$$

$$\tilde{Z}_C = \frac{-j}{0.95}\,\Omega$$

Total impedance

$$\tilde{Z} = Ze^{j\phi}$$

where $Z = 0.143\,\Omega$ and $\phi = -0.798\,\mathrm{rad}$.

$$\tilde{i} = \frac{\tilde{v}}{\tilde{Z}}$$

$$\tilde{v}_R = \tilde{i}\tilde{Z}_R = \frac{0.1e^{-j\phi}}{Z}$$

$$|\tilde{v}_R| = 0.698\,\mathrm{V}$$

$$\langle P \rangle = \frac{|\tilde{v}_R|^2}{2R} = 2.4\,\mathrm{W}$$

This is not exactly half because of the approximation $\Delta\omega \approx \gamma$ Completing the maths for the other two voltages results in the phasor diagram shown in figure 38, and for which the physical signals look like figure 39.

Finally, note that the vector sum of the three voltage phasors equals the applied voltage

$$\tilde{v} = \tilde{v}_R + \tilde{v}_L + \tilde{v}_C$$

This is required since their real components must follow Kirchhoff's voltage law

$$v(t) = v_R(t) + v_L(t) + v_C(t)$$

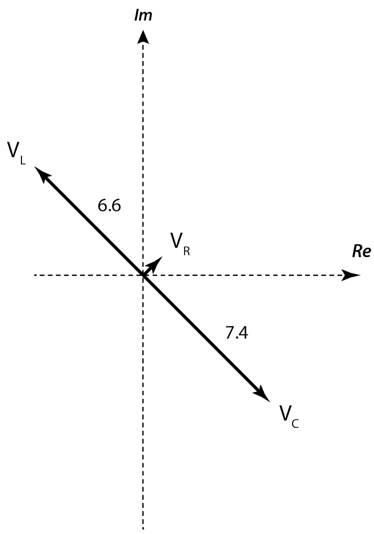

Figure 38: Phasor Diagram for LCR Circuit at Half-power

LTSpice Exercises

> Download the LTSpice schematics WorkedExample.asc and WorkedExampleBode.asc from Blackboard

This schematic WorkedExample.asc implements the circuit described in section 9.4. We can use this to verify the theory and also some of the predictions for the forced damped-oscillator given in section 4 of your Vibrations and Waves course notes.

1. Run the simulation and click a component to plot the circuit current.
2. Right-click the plot window and add a new 'plot pane'. Plot the voltage across the capacitor.
3. To plot the inductor voltage it is necessary to perform a simple calculation which can be done by right-clicking again and selecting 'add trace' then build the expression 'V(n001)-V(n002)'. Check the current and voltages agree with the calculations in section 9.4 and that the capacitor/inductor voltages are π rad out of phase.
4. Similarly, add and check the resistor voltage - this should be (in the steady-state) identical to the driving voltage.

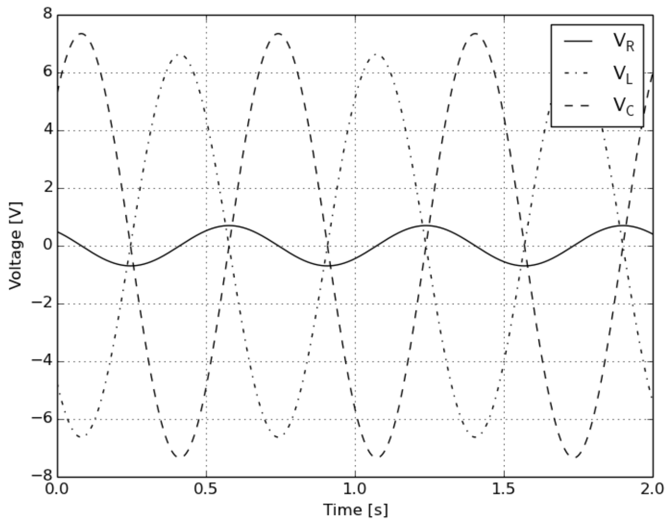

Figure 39: LCR Circuit Voltages at Half-power

5. Finally, plot the resistor power (ALT-click).
6. Right-click the voltage-source and change the driving frequency to the theoretical half-power frequency and check the results.

WorkedExampleBode.asc is setup to perform AC analysis on the same circuit.

1. Run the simulation, and click the resistor to plot the circuit current. This should appear in the form of a Bode plot.
2. Zoom in and use the cursor (click on legend) to check that the current is a maximum at ω_0 (since the impedance is real and a minimum).
3. Use the cursor to estimate the width of the peak (maximum -3 dB) - this should be approximately equal to γ. What is the value of Q for this resonance?
4. Plot also the voltage across the capacitor. Zoom in again and note that the capacitor voltage (which is proportional to the charge on the capacitor) appears to peak at a slightly different frequency.
5. This effect is accentuated by increasing the damping. Change R to $0.4\,\Omega$ and re-run.
6. According to our original formulation of the circuit differential equation (55), the amplitude of the oscillation is the charge, not the current, and the Vibrations and Waves course notes predict that the amplitude of oscillation will peak at a frequency $\omega_p = \sqrt{\omega_0^2 - \gamma^2/2}$. Check this result.

CC
13-Jan-18