Chapter 2

Differentiation

2.1 First Principles

2.2 Product, Quotient, chain-rules etc etc

2.3 Curve sketching

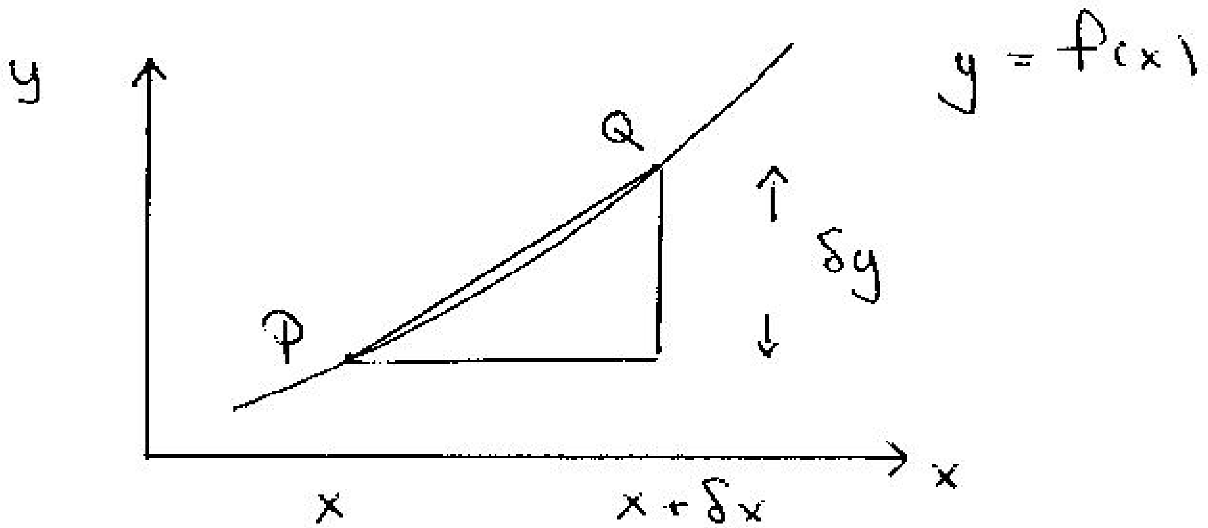
2.4 Parametric representation of curves

2.5 Polar co-ordinates

Chapter 2 : Differentiation

.1 Differentiation from First Principles

What is tangent to curve at point P?



$$\text{Gradient of Chord } PQ = \frac{f(x+\delta x) - f(x)}{\delta x} \equiv \frac{\delta y}{\delta x}$$

low, at fixed x, let $\delta x \to 0$. If limit exists,
efine

$$\boxed{\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{dy}{dx}} \quad *$$

s derivative of $f(x)$ at x.

Examples

i) $y = x^2$.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cancel{x^2} + 2x\delta x + \delta x^2 - \cancel{x^2}}{\delta x}$$

$$= \lim_{\delta x \to 0} 2x + \delta x \quad = \quad 2x$$

ii) $y = \sin x$.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

Recall $\quad \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

hence

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{2 \cos\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin \delta x/2}{\delta x/2}$$

$$= \cos x \cdot \underline{1} = \cos x$$

2.2 Useful results / techniques

·)

Product rule $\quad \dfrac{d}{dx} fg = \dfrac{df}{dx} g + f \dfrac{dg}{dx}$

uotient rule $\quad \dfrac{d}{dx} f/g = \dfrac{f'g - fg'}{g^2}$

here $f' \equiv df/dx$, $f'' \equiv d^2f/dx^2$ etc)

·) **Function of a function**

e.g. $\quad y = \ln \cos x$

write $\quad y = \ln g \quad$ and $\quad g = \cos x$

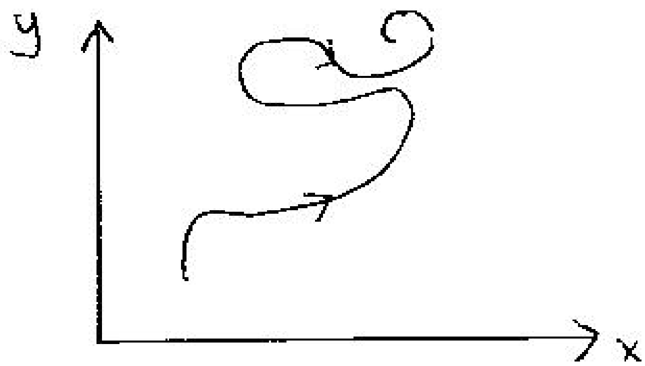
$$\frac{dy}{dx} = \frac{dy}{dg} \cdot \frac{dg}{dx} = \frac{1}{g} \cdot (-\sin x)$$

$$= -\tan x$$

chain-rule

(c) Parametric differentiation

Suppose $y = y(t)$ and $x = x(t)$ e.g. co-ordinates of moving body.



$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} \equiv \frac{\dot{y}}{\dot{x}} \quad *$$

$(\dot{y} \equiv dy/dt, \quad \ddot{y} \equiv d^2y/dt^2 \text{ etc})$

Similarly

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}$$

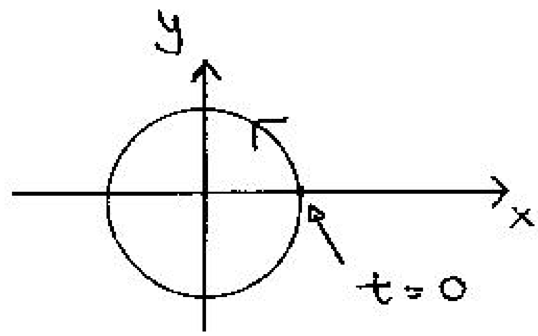
$$= \frac{dt}{dx}\cdot\frac{d}{dt}\left(\dot{y}/\dot{x}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Example

$y = \sin t$

$x = \cos t$



Have $\dot{y} = \cos t$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{\cos t}{-\sin t} = -\cot t$

$\dot{x} = -\sin t$

Similarly $\ddot{y} = -\sin t$ $\Rightarrow$ $\dfrac{d^2y}{dx^2} = \dfrac{\sin^2 t + \cos^2 t}{(-\sin t)^3} = -\dfrac{1}{y^3}$

$\ddot{x} = -\cos t$

2) Differentiation of inverse (useful ✱)

e.g. i) $y = \sin^{-1} x$

$\sin y = x \Rightarrow \cos y \cdot \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\cos y} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2 y}} \Rightarrow \boxed{y' = \dfrac{1}{\sqrt{1-x^2}}}$

ii) $y = \tan^{-1} x$

$\tan y = x \Rightarrow \sec^2 y \; y' = 1 \Rightarrow y' = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y}$

$y' = \dfrac{1}{1+x^2}$

Similarly for $y = \cos^{-1} x$, $y = \sec^{-1} x$,

$y = \sinh^{-1} x$ etc etc ...

e) Implicit Functions

Sometimes relationship btwn x and y is implicit

$$F(x, y) = 0$$

rather than explicitly $y = f(x)$.

e.g.

$$x^2 \sin y + xy = 1$$

Differentiate w.r.t. x (using product rule)

$$2x \sin y + x^2 \cos y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

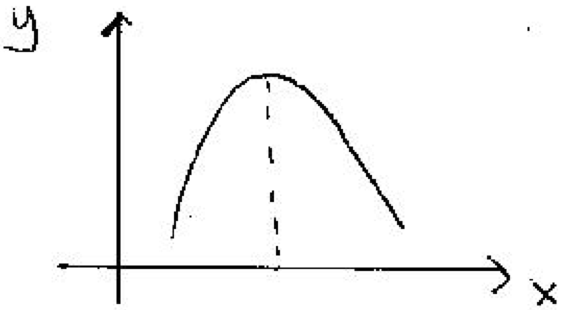
$$\frac{dy}{dx} = - \frac{(y + 2x \sin y)}{x^2 \cos y + x}$$

f) Stationary Points

At a stationary pt $\boxed{\dfrac{dy}{dx} = 0}$

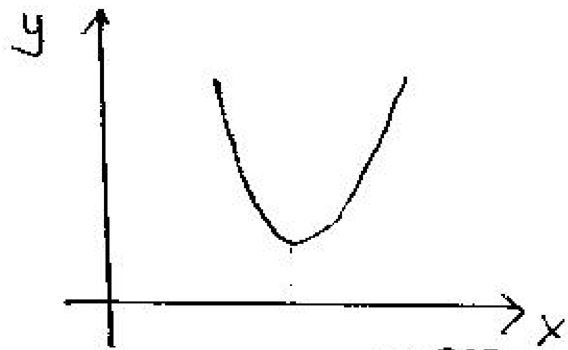
Two basic types

Maximum



$\boxed{y' = 0 \quad \text{and} \quad y'' < 0}$

Minimum



$\boxed{y' = 0\,, \quad y' > 0}$

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0$

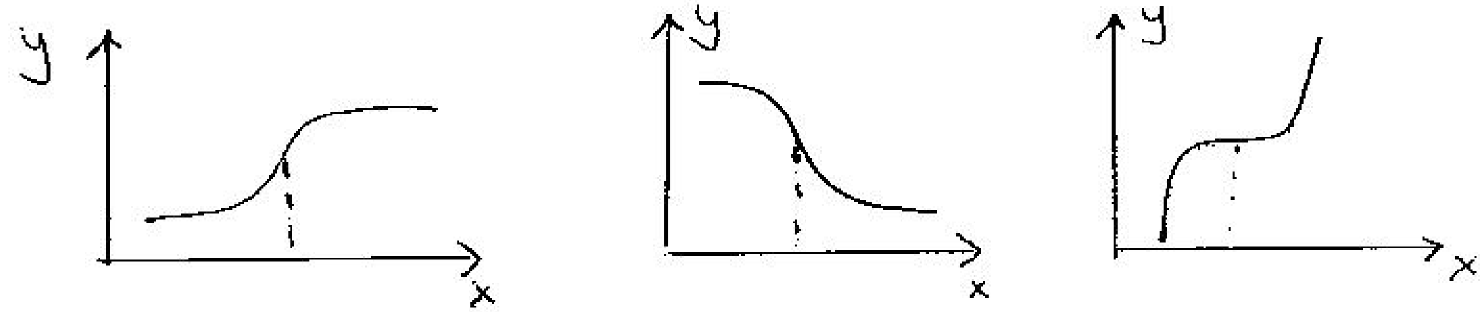
but $\dfrac{dy}{dx}$ may or not be zero.

if $y' = 0 \Rightarrow$ stationary pt of inflexion or saddle-point.

if $y' \neq 0 \Rightarrow$ non-stationary pt of inflexion)

Corresponds to a change of curvature.

Examples of points of inflexion



e.g. Find max/min^n & pts of inflexion of

$$y = x^2(x-1)$$

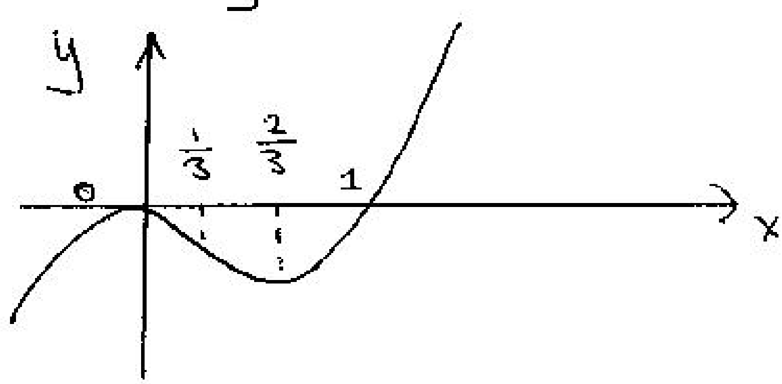
Taking derivatives

$$y' = 3x^2 - 2x = x(3x-2)$$

$$y'' = 6x - 2$$

Thus stationary pts ($y'=0$) at $x = 0, \frac{2}{3}$

- at $x = 0$, $y'' = -2 \Rightarrow$ max^n
- at $x = \frac{2}{3}$, $y'' = +2 \Rightarrow$ min^m
- $y'' = 0$ at $x = \frac{1}{3} \Rightarrow$ pt of inflex^n



1.3 Curve Sketching

Get main features by

- Examining behavior for $x \to 0, +\infty, -\infty$
- Look for symmetries (even/odd)

if $y = \dfrac{P(x)}{Q(x)}$ with P, Q polynomials

(i) zeros of P give intersections with Ox

(ii) zeros of Q give infinite discontinuities or asymptotes

Stationary pts for finer detail

Example

$$y = \frac{x(x-2)}{x-3}$$

- $y = 0$ when $x = 0, 2$

$y \sim \frac{2}{3}x$ if x small

near $x = 3$, $y \sim \dfrac{3}{x-3}$

Thus

$$y \to +\infty \quad \text{as} \quad x \to 3^+$$

$$y \to -\infty \quad \text{as} \quad x \to 3^-$$

Digression

$x \to 3^+$ means "x tends to 3 from above".

Think $x = 3.1, 3.01, 3.001 \ldots$

$x \to 3^-$ means "x tends to 3 from below"

Think $x = 2.9, 2.99, 2.999 \ldots$

(c) behaviour at large x

$$y = \frac{x(x-2)}{x-3}$$

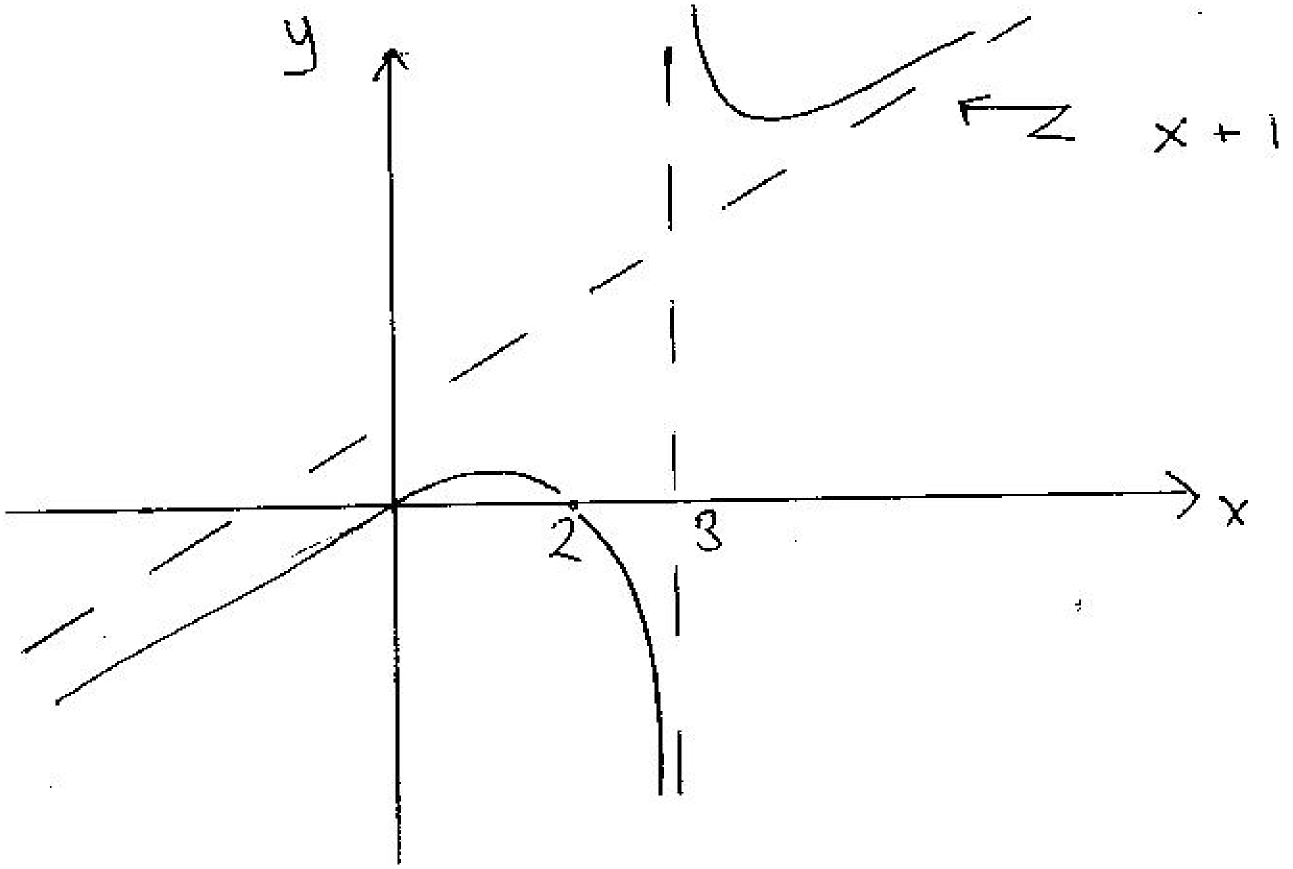
$$= \frac{x^2(1-\frac{2}{x})}{x(1-\frac{3}{x})}$$

$$= x(1-\frac{2}{x})(1-\frac{3}{x})^{-1}$$

$$= x(1-\frac{2}{x})(1+\frac{3}{x}+O(\frac{1}{x^2}))$$

$$= x(1+\frac{1}{x}+O(\frac{1}{x^2}))$$

$$= x+1+O(\frac{1}{x})$$



$$y' = \frac{x^2 - 6x + 6}{(x-3)^2} \Rightarrow \text{ stationary pts at } x = 3 \pm \sqrt{3}$$

(can already tell if max^m / min^m)

2.4 Parametric representation of curves

Given $x = x(t)$ and $y = y(t)$

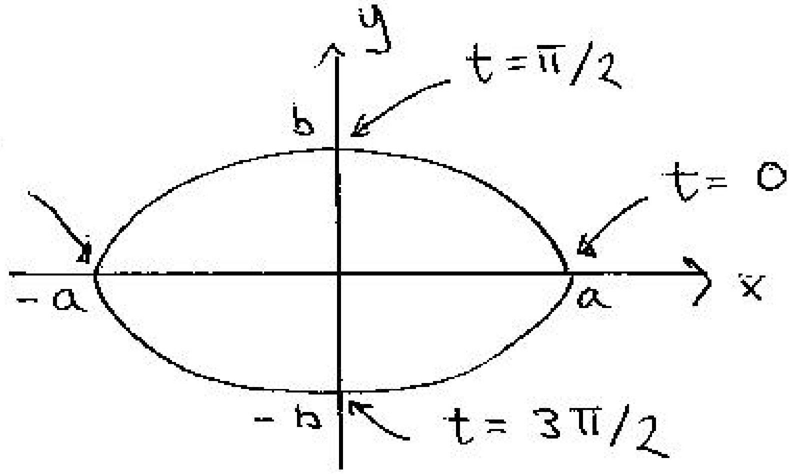
(t = time or some abstract variable)

e.g. a) $x = a\cos t$, $y = b\sin t$

In this case can elliminate t

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

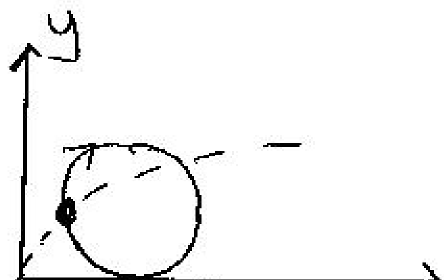
ellipse



Parametric represent^n often useful when cannot (easily) elliminate t

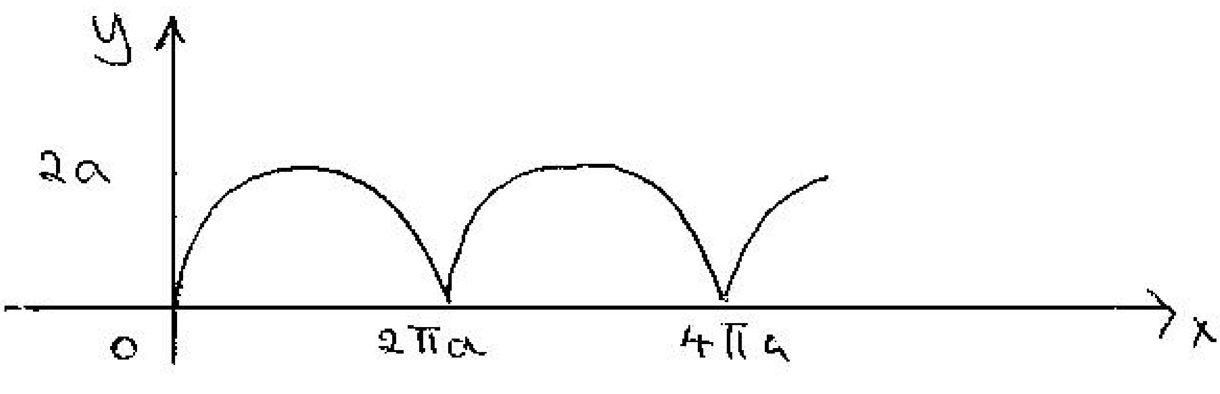
b)

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t)$$



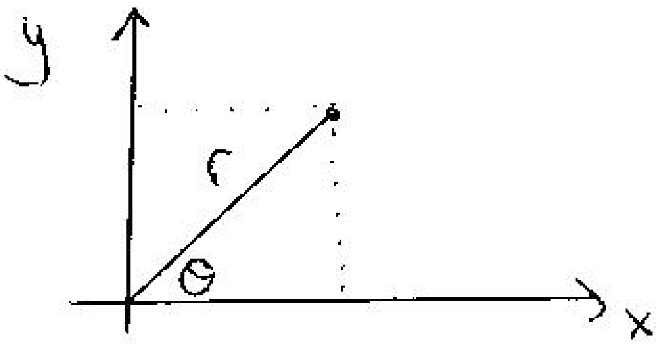
locus of point on rolling circle

Cycloid



1.5 Polar Co-ordinates

In 2D often best to use plane polars (r, θ) instead of Cartesian (x, y)



$r \equiv$ radial dist^e to origin

θ = ∠le w.r.t Ox axis

(r, θ) in terms of (x, y)

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$(0 < \theta < 2\pi)$$

(x, y) in terms of (r, θ)

$$x = r\cos\theta, \qquad y = r\sin\theta$$

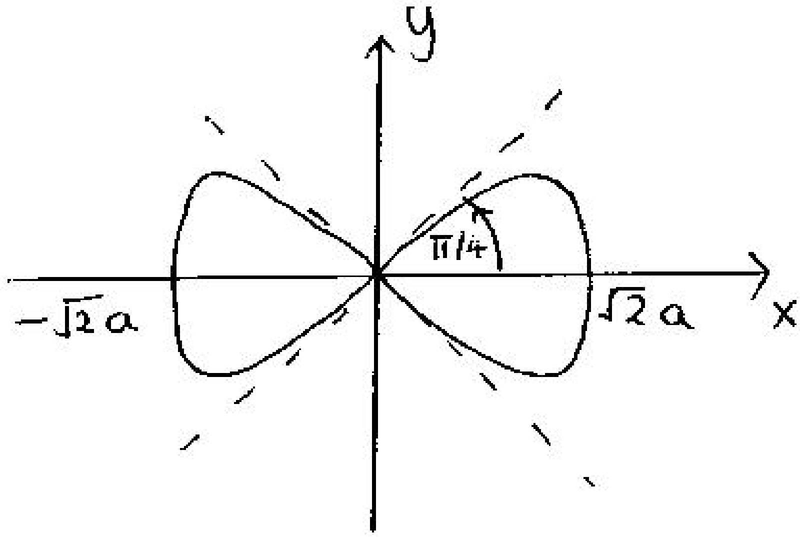
e.g. <u>Lemniscate function</u> (figure of eight)

$$r^2 = 2a^2\cos 2\theta$$

Since r real, no sol^{n}s when $\cos 2\theta < 0$

i.e. $\frac{\pi}{2} < 2\theta < \frac{3\pi}{2}$

lemniscate



Mathematical name for infinity symbol ∞.

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