Imperial College 1st Year Physics UG, 2017-18

Maths Analysis

Lecture notes

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Office hours: 11-12 Tuesday and Thursday

Tutorials: There will be 6 tutorials for the course. Three will be this term in weeks 8, 9, 10. There following three will be next term.

Example problems: During the course a number of example sheets will be handed out. Solutions will be given out one you have had a chance to attempt the problems.

Books

This course is not based directly on any one book. Recommended textbooks for the course are;

- K.E.Hirst, *Numbers, Sequences and Series* (London, Edward Arnold, 1995).
- G.Smith, *Introductory Mathematics: Algebra and Analysis* (Springer, 1998).
- M.Liebeck, *A Concise Introduction to Pure Mathematics* (Chapman and Hall, CRC, 2000).
- K.G. Binmore, *Mathematical Analysis. A Straightforward Approach* (Cambridge University Press, 1982).

The following books are less technical and even non-technical, but may be enjoyable bed-time reading:

- G.Hardy, *A Mathematician's Apology* (Cambridge University Press, 1967).
- R.Courant and H.Robbins, *What is Mathematics? An Elementary Approach to Ideas and Methods* (Oxford University Press, 1941). (See also an updated 1996 version with I.Stewart as an additional author).
- I.Stewart, *Concepts of Modern Mathematics* (Dover, 1995).

Plan for the course

- Sets and maps (or Basics):

  Sets, notation, methods of proof, Russell's paradox, maps (functions)

  Physics: clear thought, logic, computing.

- Numbers (or the Big and the Small!):

  Real numbers as infinite decimals, completeness of the reals, cardinality and countability, Cantor's proof that the reals are not countable

  Physics: infinitessimals, infinity and beyond!

- Infinite sequences (or An introduction to analysis):

  Convergence using $\epsilon - N$. Monotone and bounded sequences, subsequences. Bolzano-Weierstrass. Cauchy sequences as convergent sequences.

  Physics: numerical algorithms

- Infinite series:

  Convergence of a series, comparison test, Cauchy test, other tests, power series, Riemann reordering

  Physics: perturbation theory, physics of lattices

- Functions (and an introduction to $\epsilon - \delta$ analysis):

  Limits and continuity, differentiable functions, Taylor's theorem and series, analytic functions.

  Physics: Calculus! Approximation and understanding perturbation theory.

Some notation

In case you haven't come across this notation....

s.t.	such that
iff	if and only if
$\exists$	there exists
$\forall$	for all
$\Longrightarrow$	implies

1 Sets and Maps

1.1 Definitions and Notation

Definition (Set)
A *set* is a collection of distinct *elements* (or *members*).

Comments:

- The ordering of elements is irrelevant
- The number of elements of the same type is irrelevant; only the types of distinct elements are important

Use comma separated notation with curly braces; eg.

$$\{a, g, h, u\}\ , \quad \{5, 6\}\ , \quad \{\nabla, \triangle, \odot\}\ , \{1, 2, 3, \ldots\}$$

where ... means 'carry on in the obvious way'.

Equality: Two sets are equal if they contain the same elements.

Example:

$$\{A, B\} = \{B, A\} = \{A, B, A\} \neq \{1, 2\}$$

Note that a set may contain elements which are themselves sets,

$$\{A, \{B, C\}\} = \{\{C, B\}, A\} \neq \{B, A, C\}$$

We may name sets,

$$S_1 = \{\nabla, \triangle, \odot\}\ , \quad S_2 = \{\nabla, \triangle\}\ , \quad S_3 = \{\odot, \nabla, \triangle\}$$

and then,

$$S_1 = S_3 \neq S_2$$

Notation
We write 'x is an element of the set X' as,

$$x \in X$$

Conversely if x is not contained in the set X we write,

$$x \notin X$$

Example: using S_1 and S_2 above;

$$\nabla \in S_1 \ , \quad \odot \in S_1 \ , \quad \odot \notin S_2$$

For contrast consider a sequence;

Definition (Sequence)
A **sequence** is an ordered collection of elements, denoted with brackets, eg. (a, g, h, u, g, h) or $(2, 2, 4, 4, 6, 6, 8, 8, \ldots)$ where repetition is allowed. Then $(a, g, g, h) \neq (g, g, h, a)$ and $(a, g, g, h) \neq (a, g, h)$.

We shall later encounter sequences, but now let us return to sets.

Definition (Subset)
A set X is a *subset* of a set Y if every element in X is contained in Y.

Notation
For a subset X of Y we write;

$$X \subseteq Y$$

If X is a subset of Y, but we know it is not equal to Y (so $X \neq Y$) then we write

$$X \subset Y$$

Then there exists $y \in Y$ s.t. $y \notin X$.

If W is not a subset of Y then we write;

$$W \nsubseteq Y$$

Comment:

- Any set is a subset of itself.

Example: using the $S_{1,2,3}$ defined above;

$$S_2 \subset S_1 \ , \quad S_1 \not\subseteq S_2 \ , \quad S_1 \subseteq S_1 \ , \quad S_2 \subseteq S_2$$

Definition (Empty set)
The *empty set*, the set which contains no elements, is denoted $\emptyset$ or $\{\}$.

Comment:

- The empty set is always a subset of any set; $\emptyset \subseteq X$ for any set X.

Notation (Set builder)
We may use *set builder* notation to build sets from all elements that obey a logical condition. We denote this as,

$$X = \{x| \text{ condition on } x\}$$

where x denotes/labels elements which satisfies the specified condition.

Reads: X is the set of all elements x such that the condition on x holds.

Example: given the set of natural numbers $\mathbb{N}$, we can take the positive even numbers;

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$
$$X = \{x|x \in \mathbb{N} \,\text{and}\; x \text{ even}\} = \{0, 2, 4, 6, \ldots\}$$

Example:

$$A = \{a, b, b, c, d, e, f, h, z, z\}$$
$$B = \{d, f, e, h, h, g, i\}$$
$$C = \{x|x \in A \text{ and } x \notin B\} = \{a, b, c, z\}$$

Example:

$$D = \{d|d = 2^n \text{ and } n \in \mathbb{N}\} = \{2^0, 2^1, 2^2, 2^3, \ldots\}$$

which can also be written more compactly as,

$$D = \{2^n|n \in \mathbb{N}\}$$

A more compact notation is,

$$X = \{x \in Y \,|\, \text{condition on } x\}$$

reading X is the set of all elements of Y which satisfy the given condition.

Example: the above can be written,

$$X = \{x \in \mathbb{N} | x \text{ even}\}$$
$$C = \{x \in A | x \notin B\}$$

Definition (Finite set)
A *finite set* is a set with finitely many (distinct) elements.

Definition (Infinite set)
An *infinite set* is a set which is not finite.

Examples of finite sets;

$$A = \{1, 2, 3\}$$
$$B = \{x \in \mathbb{N} | x < 100\}$$

Some important examples of infinite sets;

Definition (Natural numbers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

Definition (Positive natural numbers)

$$\mathbb{N}^+ = \{1, 2, 3, 4, \ldots\}$$

Definition (Prime number)
A **prime** number (or 'prime') is a natural number greater than one which has exactly two divisors; 1 and itself. Denote the set $\mathbb{P} \subset \mathbb{N}^+$;

$$\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$$

Fundamental Theorem of arithmetic: Every natural number greater than one has a unique prime factorisation.

Definition (Integers)

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

Definition (Rational numbers)

$$\mathbb{Q} = \{\frac{a}{b} | a \in \mathbb{Z}\ , \quad b \in \mathbb{N}^{+}\}$$

Real numbers
$\mathbb{R}$ is the set of all numbers on the continuous real number line. We will postpone a careful definition of them until later.

Definition (Irrational numbers)
The set of real numbers which are not rational; $\mathbb{R} \setminus \mathbb{Q} = \{x \in \mathbb{R} | x \notin \mathbb{Q}\}$.

Note: An irrational number cannot be expressed as $\frac{a}{b}$ for any $a \in \mathbb{Z}$, $b \in \mathbb{N}^{+}$.

Example: In the first problem set you will prove $\sqrt{n}$ is irrational for any prime number n. Other famous irrational numbers are π, e.

Clearly,

$$\mathbb{N}^{+} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

There are some important subsets of $\mathbb{R}$ we will discuss later;

Definition (Interval): An interval A is a subset of $\mathbb{R}$, $A \subset \mathbb{R}$, such that if $x, y \in A$, and $x \leq z \leq y$ then $z \in A$.

Intervals are defined by their 'end' positions, and may include their end points of not. We use the notations;

Definition:

$$
\begin{aligned}
(a,b) &= \{x \in \mathbb{R} | a < x < b\} && \text{"Open interval"} \\
[a,b] &= \{x \in \mathbb{R} | a \leq x \leq b\} && \text{"Closed interval"} \\
(a,b] &= \{x \in \mathbb{R} | a < x \leq b\} && \text{"Half open interval"} \\
[a,b) &= \{x \in \mathbb{R} | a \leq x < b\}
\end{aligned}
$$

Also we define;

$$
\begin{aligned}
(a,\infty) &= \{x \in \mathbb{R} | a < x\} \\
[a,\infty) &= \{x \in \mathbb{R} | a \leq x\} \\
(-\infty,a) &= \{x \in \mathbb{R} | x < a\} \\
(-\infty,a] &= \{x \in \mathbb{R} | x \leq a\}
\end{aligned}
$$

1.2 Basics of proof

A proof is a **clear** and **convincing** argument. There are various methods of proof we will employ in this course.

Your should lay out a proof with the proposition (or claim), and then the argument. You should never use the result you are trying to prove in the proof itself!

The proof should end with QED (*Quod Erat Demonstrandum* - which was to be proved) or a square $\square$.

Examples of methods of proof

Proof by contradiction: You assume that what is to be proved is false, and derive a logical contradiction from this assumption.

You must state clearly in the proof that you are assuming the proposition is false for the purpose of contradiction.

Proposition: Each natural number $n > 1$ is a prime or product of primes.

Proof. Assume for contradiction that the proposition is false. Then there exists $n \in \mathbb{N}^+$, and $n > 1$ which is the smallest number such that n is not a prime nor a product of primes.

Now $n \notin \mathbb{P}$ (or it wouldn't be a counter example).

But this implies $n = x \cdot y$ for some $x, y \in \mathbb{N}^+$ and $x, y < n$ (so $x \neq 1$, $y \neq 1$).

x, y are prime or products of primes (as n was the smallest counterexample). But then we have a contradiction, as $n = x \cdot y$ is a product of primes.

Hence our assumption is false. Thus the proposition is true.

$\square$

Note that all the above statements after the assumption for contradiction are false.

Proof by induction: Suppose we have a proposition $P(n)$ about a positive number $n \in \mathbb{N}^+$ and we can show;

- $P(1)$ is true.
- Either one of the following;
  - if we choose any $n \in \mathbb{N}^+$ and assume $P(n)$ is true then $P(n+1)$ is also true.
  - if we choose any $n \in \mathbb{N}^+$ and assume $P(r)$ is true for all $r \leq n$, then $P(n+1)$ is also true.

Then it follows that $P(n)$ is true for all $n \in \mathbb{N}^+$.

Note: We were not required to prove $P(r+1)$. Only to prove it under the assumption that either $P(r)$ or $P(n)$ for all $n \leq r$ were true.

Alternatively we can replace showing $P(1)$ with showing $P(k)$ for some $k \in \mathbb{N}$, and then induction shows $P(n)$ is true for all $n \geq k$.

Proposition: For $n \in \mathbb{N}^+$ then $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Proof. (By induction)

- The proposition is true for $n = 1$. The l.h.s. $= 1$ and the r.h.s. $= \frac{1 \times 2}{2} = 1$.
- Assume the proposition holds for some $n \in \mathbb{N}^+$. Then,

$$\begin{aligned}(1 + 2 + 3 + \ldots + n) + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}\end{aligned}$$

  Hence the proposition holds also for $(n+1)$.

By induction the proposition is true for all $n \in \mathbb{N}^+$.

□

Let us reprove using induction the previous proposition we proved using contradition.

Proposition: (Again!) Each natural number $n > 1$ is a product of primes.

Proof. (By induction):

- The proposition is true for $n = 2$ as 2 is prime.

- Suppose for induction that the proposition holds for all $r \leq n$. Then consider the proposition for $n + 1$.

  If $n + 1 \in \mathbb{P}$ then the proposition holds.

  Otherwise $n + 1 \notin \mathbb{P}$. Then $n + 1 = x \cdot y$ for some $x, y \in \mathbb{N}$ and $x \neq 1$, $y \neq 1$. Then $x, y < n + 1$. But then x, y are products of primes by assumption. Thus we see $n+1$ is a product of primes so the proposition holds for $n + 1$.

By induction the proposition is true for all $n \geq 2$.

□

Some general comments:

- Try not to write $\Longrightarrow$ everywhere. Try to write the proof out in a clear and well written way - just as you would want to read it.
- Obviously do not use the result you are trying to prove.
- When proving an algebraic relation such as $x^3 - 1 = (x-1)(x^2+x+1)$, prove that the l.h.s. is equal to the r.h.s. Do not land up proving $0 = 0$!
- To contradict a proposition a single counterexample is sufficient.

  For example - *(False) Claim:* the difference between two irrational numbers is always irrational.

  This is false as $\sqrt{2} - \sqrt{2} = 0$.
- There are **necessary** conditions and **sufficient** conditions. A condition may be;
  - **Necessary and sufficient** - eg. a necessary and sufficient condition for a number to be even is its last digit must be even.
  - **Necessary but not sufficient** - eg. a necessary but not sufficient condition for a number to be divisible by 4 is its last digit must be even.
  - **Sufficient but not necessary** - eg. a sufficient but not necessary condition for a number to be divisible by 2 is its last digit is a zero.

  If we have a proposition with a 'necessary and sufficient' or 'if and only if' condition, then must be carefully to prove the two logical statements.

1.3 Properties of sets

Definition (Cardinality)
The *Cardinality* of a finite set is the number of elements it contains; denoted $|X|$ for the cardinality of a set X.

Definition (Power set)
The *power set* of a set X is the set of all subsets of X and is denoted 2^X;

$$2^X = \{A|A \subseteq X\}$$

Reads: 2^X is the set of all elements A such that A is a subset of X.

Example:

$$\begin{aligned} A &= \{1, 2, 3\} \\ 2^A &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \end{aligned}$$

Theorem 1.1 (Cardinality of the power set)**.** *The power set of a finite set X has cardinality $|2^X| = 2^{|X|}$, hence the notation.*

Proof. Choose some ordering of the elements in the finite set $X = \{x_1, x_2, x_3, \ldots, x_n\}$ where $n = |X|$. Then a subset is specified by the n-digit binary number

$$p_1 p_2 \ldots p_n \ , \quad p_i \in \{0, 1\}$$

where $p_i = 0$ means x_i is not in the subset, and $p_i = 1$ means x_i is included in the subset.

Thus the number of subsets is equal to the number of n-digit binary numbers, i.e., 2^n.

□

Definition (Union, Intersection, Set Difference)
Let X and Y be sets. Then;

- The **union** of X and Y, denoted $X \cup Y$, is;
$$X \cup Y = \{z | z \in X \text{ or } z \in Y\}$$
- The **intersection** of X and Y, denoted $X \cap Y$, is;
$$X \cap Y = \{z | z \in X \text{ and } z \in Y\}$$
- The **set difference** of X from Y, denoted $X \setminus Y$, is;
$$X \setminus Y = \{z | z \in X \text{ and } z \notin Y\}$$

We say two sets X and Y are **disjoint** iff $X \cap Y = \emptyset$.

Some properties:

- Union and intersection are *associative*: for any sets A, B, C then;
$$A \cup (B \cup C) = (A \cup B) \cup C$$
so we simply write this as $A \cup B \cup C$. Likewise for intersection.
- For any set A then; $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$

Examples: for the sets $A = \{1, 2, 3\}$ as before, $B = \{2, 6\}$ and $C = \{0, 1, 2, 3, 4\}$ then,

$$A \cup B = \{1, 2, 3, 6\}$$
$$A \cap B = \{2\}$$
$$A \setminus B = \{1, 3\}$$
$$B \setminus A = \{6\}$$
$$A \setminus C = \emptyset$$

and,

$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$
$$A \setminus \emptyset = A$$

Definition (Cartesian product)
The **Cartesian product** of two sets X and Y, denoted $X \times Y$, is the set of **ordered** pairs (or sequences of length two) of elements, the first coming from X and second from Y.

$$X \times Y = \{(x, y) | x \in X\ ,\ y \in Y\}$$

Comments:

- $X \times Y \neq Y \times X$ unless $X = Y$.
- For finite sets $|X \times Y| = |X| \times |Y|$.

Examples: for the sets A, B and C above,

$$A \times B = \{(1,2),(1,6),(2,2),(2,6),(3,2),(3,6)\}$$
$$B \times A = \{(2,1),(6,1),(2,2),(6,2),(2,3),(6,3)\}$$

Note that $A \neq B$.

This generalizes in the natural way for the product of finitely many multiple sets,

$$X_1 \times X_2 \times \ldots \times X_n = \{(x_1, x_2, \ldots, x_n) | x_1 \in X_1\ ,\ x_2 \in X_2\ ,\ \ldots\ ,\ x_n \in X_n\}$$

where the elements $(x_1, x_2, \ldots x_n)$ are sequences. Likewise for an infinity of sets, so each X_n is a set for all $n \in \mathbb{N}^+$ we may define;

$$X_1 \times X_2 \times \ldots = \{(x_1, x_2, \ldots) | x_n \in X_n \forall n \in \mathbb{N}^+\}$$

Taking the Cartesian product of a set X with itself n-times we use the notation; $X^n = X \times X \times \ldots \times X$ (for n-factors on the r.h.s.)

Examples:

- Two dimensional coordinates (x, y) where $x, y \in \mathbb{R}$ are denoted $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
- n-coordinates are elements of $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$

1.4 Russell's Paradox

A set is a collection of elements.

- Can these elements be anything we like?
- Can the collection be arbitrary?

An example of a 'wild' set is the **set of all sets**:

$$S = \{x|x \text{ is a set}\}$$

Problems in the definitions of sets arise when we take the members of sets to be themselves sets in an unrestricted manner.

Russell's Paradox: Define the set of all sets that do not have themselves as an element:

$$R = \{x|x \text{ is a set },\ x \notin x\}$$

Now R is a set whose elements are sets. Is it an element of itself?

- Suppose the answer were yes, so $R \in R$. This would imply $R \notin R$ by construction of R.
- If the answer were no, so $R \notin R$, then by construction this would imply $R \in R$.

Thus either answer, yes or not, leads to contradiction.

Russell's paradox was removed by introducing set axioms that don't allow the sets like S and R - for example the Zermelo-Fraenkel axioms.

These include a 'foundation' axiom; All non-empty sets X have a member Y such that X and Y are disjoint sets. This rules out sets containing themselves, and sets like $\{\{\{\{\ldots\}\}\}\}$.

1.5 Maps (functions)

Definition (Map/Function)
A **map** from a set D to a set T is an assignment of each element of D to some element of T.

- D is the **domain**
- T is the **target** or **codomain**
- If we label the map - say f - we use the notations;

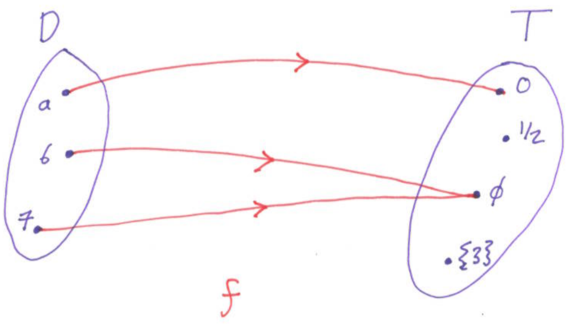
$$f: \quad D \to T$$

or if we are more explicit about specifying the map,

$$\begin{aligned} f: \quad & D \to T \\ & x \to f(x) \end{aligned}$$

where $x \in D$ and $f(x) \in T$.

- $f(x)$ is the **image** of x under the map f
- x is the **pre-image** of $f(x)$ under the map f



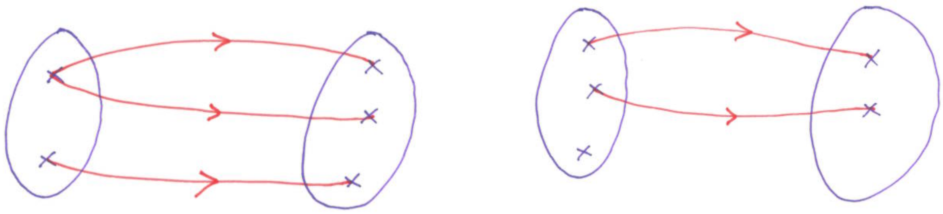
Example: $D = \{a, 6, 7\}$ to $T = \{0, \frac{1}{2}, \emptyset, \{3\}\}$. Consider a map $g : D \to T$ defined as,

$$\begin{aligned} g(a) &= 0 \\ g(6) &= \emptyset \\ g(7) &= \emptyset \end{aligned}$$

Example:

$$h: \quad \mathbb{R} \to \mathbb{R}$$
$$x \to h(x) = x + 1$$

Two examples which are not maps;



Definition (Identity map)
For any domain we may define the **identity map**, notated id_D, which 'does nothing';

$$id_D : D \to D$$
$$x \to x$$

Definition (Image)
The **image** (or **range**) of a map $f : D \to T$ is the set of elements of T that those of D are mapped to.

$$\text{Image}(f) = \{y \in T |\, \exists\, x \in D \text{ s.t. } f(x) = y\}$$

This is sometime written $f(D) = \text{Image}(f)$.

Example: in the above maps then Image$(g) = \{0, \emptyset\}$ and Image$(h) = \mathbb{R}$.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \sqrt{x}$, is **not** a map as $x = -1$ doesn't have an image in $\mathbb{R}$.
   However,
$$\begin{aligned} h : [0, \infty) &\to \mathbb{R} \\ x &\to \sqrt{x} \end{aligned}$$
   is a well defined map. Note that Image$(h) = [0, \infty)$

2. $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = 1/x$, is **not** a map as $f(0)$ isn't well defined.
   However,
$$\begin{aligned} j : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\to \frac{1}{x} \end{aligned}$$
   is a well defined map. Note that Image$(j) = \mathbb{R} \setminus \{0\}$.

3. $f : \mathbb{R} \to \mathbb{R}$, with $f(x)$ defined so that $f(x) = \tan(x)$, is **not** a map. The map is not well defined at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$.
   However,
$$\begin{aligned} k : (-\frac{\pi}{2}, +\frac{\pi}{2}) &\to \mathbb{R} \\ x &\to \tan x \end{aligned}$$
   is a well defined map.

4. $f : \mathbb{R} \to \mathbb{R}$, with $f(x)$ defined so that $\sin(f(x)) = x$, is **not** a map. Firstly there are many values z such that $\sin z = 0$ so there is no unique specification of a map. Secondly consider $|x| > 1$ which doesn't get mapped anywhere.
   However,
$$\begin{aligned} l : [-1, +1] &\to [-\frac{\pi}{2}, +\frac{\pi}{2}] \\ x &\to \arcsin x \end{aligned}$$
   is a well defined map. Note that Image$(l) = [-\frac{\pi}{2}, +\frac{\pi}{2}]$.

Definition (Composition)
Given maps $f : X \to Y$ and $g : Y \to Z$ then we define the **composition** of f with g as,

$$\begin{aligned} g \cdot f : X &\to Z \\ x &\to g \cdot f(x) = g(f(x)) \end{aligned}$$

Note: $g \cdot f(x)$ means act first with f, then act with g on the result.

Proposition 1.1. *Let $f : X \to Y, g : Y \to Z$ and $h : Z \to W$ be maps. Then,*

$$h \cdot (g \cdot f)(x) = (h \cdot g) \cdot f(x)$$

This means we **unambiguously** *write the composition of these 3 maps as $h \cdot g \cdot f$.*

Proof. Consider LHS:

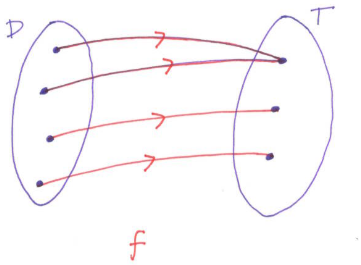
$$h \cdot (g \cdot f)(x) = h(g \cdot f(x)) = h(g(f(x)))$$

Consider RHS:

$$(h \cdot g) \cdot f(x) = (h \cdot g)(f(x)) = h(g(f(x)))$$

Hence the LHS = RHS. □

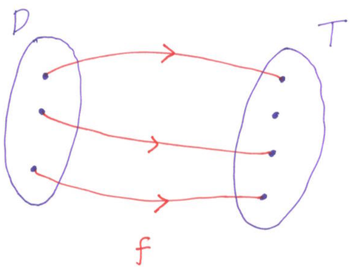
Definition (Surjective)
A map $f : D \to T$ is **surjective** (or **onto**) if for every $y \in T$ there exists an $x \in D$ s.t. $f(x) = y$.



No target element is 'missed out'

Note: for a surjection $f : D \to T$ then Image$(f) = T$.

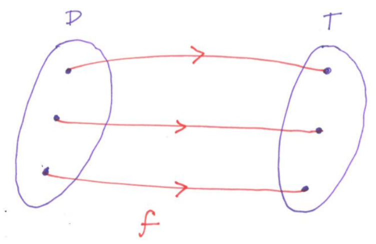
Definition (Injective)
A map $f : D \to T$ is **injective** (or **one-to-one**) if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in D$.



Alternative definition (Injective)
A map $f : D \to T$ is **injective** (or **one-to-one**) if $f(x) = f(y)$ implies $x = y$ for all $x, y \in D$.

Every domain element is uniquely paired with one in the target.

Definition (Bijective)
A map $f : D \to T$ is **bijective** if it is both **injective** and **surjective**.



Each domain element is uniquely paired with a target element and vice versa.

Definition (Inverse)
Given a **bijective map** $f : D \to T$, then the inverse map, denoted f^{-1}, is defined by,

$$\begin{aligned} f^{-1} : T &\to D \\ y &\to x(y) \end{aligned}$$

such that $y = f(x)$. Such a map exists because f is surjective, and is unique because f is an injection. Then,

$$f^{-1} \cdot f = id_D$$
$$f \cdot f^{-1} = id_T$$

Proposition 1.2. *Let $f : A \to B$ be a map between* **finite** *sets A and B. Then,*

1. *If f is a surjection then $|A| \geq |B|$*
2. *If f is an injection then $|A| \leq |B|$*
3. *If f is a bijection then $|A| = |B|$*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ so $n = |A|$. Let,

$$R = \text{Image}(f) = \{b \in B | \exists\, a_i \in A \text{ s.t. } f(a_i) = b\}$$

Hence we may write, $R = \{f(a_1), f(a_2), \ldots, f(a_n)\} \subseteq B$ although generally some $f(a_i)$ will be equal, and hence the list for R may have repeated elements. Hence we have $|R| \leq |A| = n$ and $|R| \leq |B|$. Now,

1. If f is a surjection then $R = B$ and so $|B| \leq n$. Therefore $|B| \leq |A|$.

2. If f is an injection then $f(a_i) \neq f(a_j)$ for $i \neq j$, and so $|R| = |A| = n$. Then, $|A| \leq |B|$.

3. If f is a bijection then both $|B| \leq |A|$ and $|A| \leq |B|$ must be true which implies $|A| = |B|$.

$\square$

Corollary 1.1. *The Pigeonhole Principle*
If you have N pigeons and M pigeonholes and $N > M$, some pigeon hole must have more than one pigeon in!

[If $f : A \to B$ for finite sets A, B s.t. $|A| > |B|$ then f is not an injection.]