4 Series

The following are examples of **series** or **series sums**.

$$\begin{aligned} \sqrt{2} &= 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} + \dots \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \quad \text{if} \quad |x| < 1 \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned} \tag{92}$$

Definition (Series):
A **series** (or **series sum**) is the infinite sum of a real sequence (a_n). It is denoted, $\sum_{k=1}^{\infty} a_k$. We term the a_n the summands.

Note that we may define $b_n = a_{n-N+1}$ and then,

$$\sum_{k=N}^{\infty} b_k = \sum_{k=1}^{\infty} a_k \tag{93}$$

so we may write a sum 'starting' at any number.
For example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{if} \quad |x| < 1 \tag{94}$$

The value of a series is defined by our notion of convergence.

Definition (Partial sum):
The n'th **partial sum** of a series $\sum_{k=1}^{\infty} a_k$ is denoted S_n and is the sum of the first n terms, ie.

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \tag{95}$$

Example: for the series $\sqrt{2} = 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} +$ then $S_3 = 1.41$.

The partial sums form a sequence (S_n).

4.1 Convergent series

Definition (value of a series):
If for a series $\sum_{k=1}^{\infty} a_k$ the sequence of partial sums, (S_n), converges so $S = \lim_{n\to\infty} S_n$, then we say the **series converges** (or the **series is convergent**), and its value is S, so,

$$S = \sum_{k=1}^{\infty} a_k \tag{96}$$

If (S_n) does not converge then the series is **divergent**.

Recall from the example sheets you proved (by induction);

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \tag{97}$$

Proposition 4.1. *Geometric series*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if} \quad |x| < 1 \tag{98}$$

Proof. Define the partial sums,

$$S_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - x^{n+1} \cdot \frac{1}{1-x} \tag{99}$$

by the previous exercise. Now if $|x| < 1$ then $x^{n+1} \to 0$ as $n \to \infty$, and hence $\frac{x^{n+1}}{1-x} \to 0$ in this limit leaving,

$$S_n \to \frac{1}{1-x} \quad \text{as} \quad n \to \infty \tag{100}$$

□

Proposition 4.2. *If $x = \sum_{k=1}^{\infty} x_k$ and $y = \sum_{k=1}^{\infty} y_k$ are convergent series, then,*

$$a \cdot x + b \cdot y = \sum_{k=1}^{\infty} (a \cdot x_k + b \cdot y_k) \tag{101}$$

for any $a, b \in \mathbb{R}$.

Proof. Consider the partial sums $S_n = \sum_{k=1}^{n} x_k$ and $T_n = \sum_{k=1}^{n} y_k$. Then these converge, $S_n \to x$ and $T_n \to y$. Now consider,

$$U_n = \sum_{k=1}^{n} (ax_k + by_k) \tag{102}$$

so $U_n = aS_n + bT_n$ for any $n \in \mathbb{N}^+$. From our previous results on sequences, $U_n \to ax + by$ as $n \to \infty$. $\square$

Convergence of a series depends only on the **tail** of the sequence of partial sums (not the first terms - or **head**). For example,

$$5 + 19 - 2^{29} + 1 + \frac{1}{2} - 15^{2^{2^{2^2}}} + \underbrace{\qquad \cdots\cdots \qquad}_{10^{2000000} \text{ terms}} + 3 + \sum_{k=1}^{\infty} \frac{1}{2^k} \tag{103}$$

is a convergent series.

Proposition 4.3. *The series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *is divergent.*

Proof. Firstly we see for $n > 0$;

$$
\begin{aligned}
S_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\
&\quad + \left(\frac{1}{9^p} + \frac{1}{10^p} + \dots + \frac{1}{16^p}\right) + \\
&\quad \vdots \\
&\quad + \left(\frac{1}{2^{n-1}+1} + \dots + \frac{1}{2^n}\right) \\
&\qquad\quad 2^{n-1} \text{ terms} \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\
&\quad + \left(\frac{1}{16^p} + \frac{1}{16^p} + \dots + \frac{1}{16^p}\right) + \\
&\quad \vdots \\
&\quad + \left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = 1 + \frac{n}{2} \\
&\qquad\quad n \text{ terms}
\end{aligned} \tag{104}
$$

Thus the partial sums (S_{2^n}) are unbounded from above as $n \to \infty$. Thus the sequence (S_n) is also unbounded above and diverges as $n \to \infty$.

□

Proposition 4.4. *If a series* $\sum_{k=1}^{\infty} a_k$ *has positive terms,* $a_n \geq 0$ *and* (S_n) *is bounded above then the series converges.*

Proof. If $a_n \geq 0$ the (S_n) is an increasing sequence. An increasing sequence that is bounded above converges.

□

If we can show (S_n) is increasing and bounded above then we learn the series converges. However we don't learn what value it converges to.

Proposition 4.5. *The series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *is convergent for* $p > 1$. *(This is the Riemann zeta function* $\zeta(p)$*).*

Proof. Firstly we see for $n > 0$;

$$\begin{aligned}
\sum_{k=1}^{2^n-1} \frac{1}{k^p} &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots \\
&\quad + \left(\frac{1}{8^p} + \frac{1}{9^p} + \dots + \frac{1}{15^p}\right) + \\
&\quad \vdots \\
&\quad + \underbrace{\left(\frac{1}{(2^{n-1})^p} + \dots + \frac{1}{(2^n - 1)^p}\right)}_{2^{n-1} \text{ terms}} \\
&\leq 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \dots \\
&\quad + \left(\frac{1}{8^p} + \frac{1}{8^p} + \dots + \frac{1}{8^p}\right) + \\
&\quad \vdots \\
&\quad + \left(\frac{1}{(2^{n-1})^p} + \dots + \frac{1}{(2^{n-1})^p}\right) \\
&= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \dots + \frac{2^{n-1}}{2^{p(n-1)}} \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \dots + \frac{1}{2^{(n-1)(p-1)}} \\
&= \frac{1 - \left(\frac{1}{2^{(p-1)}}\right)^n}{1 - \frac{1}{2^{(p-1)}}}
\end{aligned} \tag{105}$$

Now since $p > 1$ then $\frac{1}{2^{(p-1)}} < 1$ so,

$$\sum_{k=1}^{2^n-1} \frac{1}{k^p} \leq \frac{1}{1 - \frac{1}{2^{(p-1)}}} \tag{106}$$

Now since $2^n - 1 \geq n$ for any $n \in \mathbb{N}^+$ then,

$$S_n \leq S_{2^n-1} \leq \frac{1}{1 - \frac{1}{2^{(p-1)}}} \tag{107}$$

Thus the sequence (S_n) is bounded. Since $a_n > 0$ (S_n) is increasing. Hence it must converge.

□

Lemma 4.1. *(Simple comparison test I)*
Let $\sum_{k=1}^{\infty} b_k$ *be a convergent series such that* $0 \leq b_k$. *Then the series* $\sum_{k=1}^{\infty} a_k$ *converges if* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}^+$.

Proof. Consider the partial sums,

$$S_n = \sum_{k=1}^{n} a_k \ , \quad T_n = \sum_{k=1}^{n} b_k \tag{108}$$

Then both (S_n) and (T_n) are both **increasing** sequences and $S_n \leq T_n$ for all $n \in \mathbb{N}^+$.

Now since $\sum_{k=1}^{\infty} a_k$ converges there exists $T \in \mathbb{R}$ such that $T_n \to T$ as $n \to \infty$. The sequence (T_n) is bounded by T and hence so is (S_n).

So (S_n) is an increasing sequence that is bounded above, so it converges. □

Lemma 4.2. *(Simple comparison test II) Let* $\sum_{k=1}^{\infty} b_k$ *be a divergent series such that* $0 \leq b_k$. *Then the series* $\sum_{k=1}^{\infty} a_k$ *diverges if* $0 \leq b_k \leq a_k$ *for all* $k \in \mathbb{N}^+$.

The proof is left as an exercise.

4.2 Tests of convergence

We have seen above in the case where $a_n \geq 0$ and (S_n) is bounded then the series converges. Suppose that the summands a_n are not positive - what can we say more generally about convergence? In fact there are a number of 'tests' of convergence.

Since we don't know whether a given series converges, and what the limit of (S_n) is, a powerful method to deal with series is observing convergence is equivalent to (S_n) being a Cauchy sequence. Recall Cauchy sequences are useful as they are equivalent to convergent sequences, but do not explicitly refer to the limit.

Proposition 4.6. *(Cauchy convergence test)*
A series is convergent if and only if for any $\epsilon > 0$ there exists N such that,

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon \tag{109}$$

for all $n \geq m > N$.

Proof. A series is convergent iff the partial sums (S_n) are a convergent sequence. Hence it is convergent iff (S_n) is a Cauchy sequence.

Let $\epsilon > 0$. If (S_n) is Cauchy there exists N such that for all $n, m' > N$ such that $|S_n - S_{m'}| < \epsilon$. Take $n > m'$, then,

$$S_n - S_{m'} = \sum_{k=m'+1}^{n} a_k \tag{110}$$

So taking $m = m' + 1$, then for all $n \geq m > N$,

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon \tag{111}$$

□

Corollary 4.1. *A necessary, but not sufficient, condition for convergence is;*

$$|a_n| \to 0 \quad \text{as} \quad n \to \infty \tag{112}$$

Note: If (a_k) does not converge to zero, the series $\sum_{k=1}^{\infty} a_k$ does not converge.

Proof. That this is necessary for a convergent series follows directly from proposition 4.6 above taking the case $n = m$.

That it is not sufficient is shown by the example $\sum_{k=1}^{\infty} \frac{1}{k}$, which as $a_n \to 0$ as $n \to \infty$ but is divergent.

□

Example: The series $\sum_{k=0}^{\infty} (-1)^k$ is divergent since $a_n \not\to 0$ as $n \to \infty$.

In the case $a_n \geq 0$, we required boundedness of (S_n) to show convergence. However, in an 'alternating' series where the signs of a_n alternate, convergence is automatic if the norms $|a_n|$ decrease with n.

Lemma 4.3. *(Alternating series test)*
Let (b_k) be a decreasing sequence such that $b_k \to 0$ as $k \to \infty$. Then the 'alternating' series,

$$\sum_{k=0}^{\infty} (-1)^{k-1} b_k = b_1 - b_2 + b_3 - b_4 + \dots \tag{113}$$

converges.

Note: $b_k \geq 0$

Proof. We observe $0 \leq b_n - b_{n+1}$ for all n as (b_n) is decreasing.

Consider any $n > m$ and the sum; $I = b_{m+1} - b_{m+2} + b_{m+3} - \dots b_n$.

If the number of terms is even we may write;

$$I = b_{m+1} - (b_{m+2} - b_{m+3}) - \dots - (b_{n-2} - b_{n-1}) - b_n \leq b_{m+1} \tag{114}$$

or write,

$$I = (b_{m+1} - b_{m+2}) + (b_{m+3} - b_{m+4}) + \dots + (b_{n-1} - b_n) \geq 0 \tag{115}$$

On the other hand if the number of terms is odd;

$$I = b_{m+1} - (b_{m+2} - b_{m+3}) - \dots - (b_{n-1} - b_n) \leq b_{m+1} \tag{116}$$

or write,

$$I = (b_{m+1} - b_{m+2}) + (b_{m+3} - b_{m+4}) + \ldots + (b_{n-2} - b_{n-1}) + b_n \geq 0 \quad (117)$$

Hence we see for any $n > m$;

$$0 \leq b_{m+1} - b_{m+2} + b_{m+3} - \ldots b_n \leq b_{m+1} \quad (118)$$

Let $\epsilon > 0$. Since $b_n \to 0$ as $n \to \infty$ there exists N such that for all $k > N$ then $|b_k| = b_k < \epsilon$.

So for any $n > m > N$ then we have,

$$\begin{aligned} |S_n - S_m| &= |(b_1 - b_2 + b_3 - \ldots b_n) - (b_1 - b_2 + b_3 - \ldots b_m)| \\ &= |b_{m+1} - b_{m+2} + b_{m+3} - \ldots b_n| \leq b_{m+1} < \epsilon \end{aligned} \quad (119)$$

Thus (S_n) is Cauchy and converges.

□

Corollary 4.2. *The sequence,*

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} \quad (120)$$

converges.

Note: In fact this converges to $\ln 2$.

Lemma 4.4. *(Comparison test)*
Let $\sum_{k=1}^{\infty} b_k$ be a convergent series such that $0 \leq b_k$. Then the series $\sum_{k=1}^{\infty} a_k$ converges if there exists $N \in \mathbb{N}^+$ such that $|a_k| \leq b_k$ for all $k \geq N$.

Note: Before we only had this if $a_k > 0$. Also this now only applies to the tail of the series.

Proof. Let $\epsilon > 0$. Take S_n and T_n to be the partial sums for $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ respectively. Since $\sum_{k=1}^{\infty} b_k$ converges, then (T_n) is Cauchy so there exists M such that for any $n > m > M$,

$$\left| \sum_{k=m}^{n} b_k \right| = \sum_{k=m}^{n} b_k < \epsilon \quad (121)$$

Let may choose $M \geq N$. Now,

$$\begin{aligned} |S_n - S_m| &= |a_{m+1} + a_{m+2} + \ldots + a_n| \\ &\leq |a_{m+1}| + |a_{m+2}| + \ldots + |a_n| \\ &\leq b_{m+1} + b_{m+2} + \ldots + b_n = \sum_{k=m}^{n} b_k < \epsilon \end{aligned} \tag{122}$$

Hence (S_n) is Cauchy, so $\sum_{k=1}^{\infty} a_k$ converges.

□

Definition (absolute convergence):
A series $\sum_{k=1}^{\infty} a_k$ **converges absolutely** (or is **absolutely convergent**) if $\sum_{k=1}^{\infty} |a_k|$ converges.

Example: The sum $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ converges but is not absolutely convergent.

Clearly this example illustrates a convergent series need not be absolutely convergent. The converse is true however.

Lemma 4.5. *An absolutely convergent series is convergent.*

Proof. We assume $\sum_{k=1}^{\infty} |a_k|$ converges. Take $b_k = |a_k|$, and apply the Comparison test, comparing $\sum_{k=1}^{\infty} a_k$ to the convergent $\sum_{k=1}^{\infty} b_k$, $b_k \geq 0$.

Since $|a_k| = b_k$ this implies $\sum_{k=1}^{\infty} a_k$ converges.

□

Lemma 4.6. *(The root test)*
Let $\sum_{k=1}^{\infty} a_k$ *be a series and let* $x_n = |a_n|^{\frac{1}{n}}$ *for all* $n \in \mathbb{N}^+$. *Suppose that* (x_n) *is a convergent sequence. Then if;*

- $x_n \to x$ *as* $n \to \infty$ *and* $x > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *is divergent.*
- $x_n \to x$ *as* $n \to \infty$ *and* $x < 1$, *then* $\sum_{k=1}^{\infty} a_k$ *is convergent.*
- $x_n \to 1$ *as* $n \to \infty$ *then the test is inconclusive.*

Proof. Let us consider the cases.

Case 1: Suppose $x_n \to x$ as $n \to \infty$ and $x > 1$. Then choose $\rho \in \mathbb{R}$ such that $1 < \rho < x$ and let $\epsilon = x - \rho$.

Now (x_n) is a convergent sequence, so there exists $N \in \mathbb{N}^+$ such that for $n > N$ then $|x_n - x| < \epsilon$.

Hence for all $n > N$ then $x_n > \rho$, and so $|a_n| > \rho^n > 1$. Thus (a_n) does not tend to zero, and so the series cannot converge.

Case 2: Suppose $x_n \to x$ as $n \to \infty$ and $x < 1$. Choose $r \in \mathbb{R}$ such that $0 < x < r < 1$ and let $\epsilon = r - x$.

Then there exists $N \in \mathbb{N}^+$ such that for $n > N$ then $|x_n - x| < \epsilon$ and hence $x_n < r$.

Thus for all $n > N$ then $|a_n| < r^n$.

Then by the comparison test, comparing $\sum_{n=1}^{\infty} a^n$ to $\sum_{n=1}^{\infty} r^n$ (which converges since $0 < r < 1$) for the terms $n > N$ we see convergence.

Case 3: Consider two series;

- $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Note $\left(\frac{1}{k}\right)^{\frac{1}{k}} \to 1$ as $k \to \infty$.
- $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ converges. Note $\left|\frac{(-1)^{k-1}}{k}\right|^{\frac{1}{k}} = \left(\frac{1}{k}\right)^{\frac{1}{k}} \to 1$ as $k \to \infty$.

Thus both convergence and divergence is possible for $|a_k|^{1/k} \to 1$.

□

Lemma 4.7. *(The ratio test)*
Let $\sum_{k=1}^{\infty} a_k$ *be a series and let* $y_n = \left|\frac{a_{n+1}}{a_n}\right|$, *for all* $n \in \mathbb{N}^+$. *Then if,*

- $y_n \to y$ *as* $n \to \infty$ *and* $y > 1$ *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*
- $y_n \to y$ *as* $n \to \infty$ *and* $y < 1$ *then* $\sum_{k=1}^{\infty} a_k$ *is convergent.*
- $y_n \to 1$ *as* $n \to \infty$ *then the test is inconclusive.*

Proof. The proof is very similar to the root test.

Case 1: Suppose $y_n \to y$ as $n \to \infty$ and $y > 1$. Then choose $\rho \in \mathbb{R}$ such that $1 < \rho < y$. Then there exists $N \in \mathbb{N}^+$ such that for $n > N$ then $y_n > \rho$.

Hence for all $n > N$ then,

$$\left| \frac{a_{n+1}}{a_n} \right| > \rho \tag{123}$$

so that, $|a_{n+1}| > |a_n|$. Thus for $n > N$ the sequence $(|a_n|)$ is strictly increasing, so $|a_{n+1}| > |a_{N+1}|$ for all $n > N$, and so the sequence (a_n) cannot tend to zero, and so the series cannot converge.

Case 2: Suppose $y_n \to y$ as $n \to \infty$ and $y < 1$. Then choose $r \in \mathbb{R}$ such that $y < r < 1$. Then there exists $N \in \mathbb{N}^+$ such that for $n > N$ then $y_n < r$.

Then, $y_{N+1} = \left| \frac{a_{N+2}}{a_{N+1}} \right| < r$ so that $|a_{N+2}| < r|a_{N+1}|$.

Similarly, $|a_{N+3}| < r|a_{N+2}| < r^2|a_{N+1}|$, and so on, so we see,

$$|a_k| < r^{k-(N+1)}|a_{N+1}| \text{ for } \quad k > N + 1 \tag{124}$$

Then by the strong comparison test, we see $\sum_{k=1}^{\infty} a_k$ converges by comparison to the convergent $\sum_{k=1}^{\infty} \left(r^k \frac{|a_{N+1}|}{r^{N+1}} \right) = \frac{|a_{N+1}|}{r^{N+1}} \sum_{k=1}^{\infty} r^k$ for the terms $k > N+1$. (Note $0 < r < 1$).

Case 3: Consider two series;

- $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
- $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ converges.

Both have $|a_{n+1}/a_n| \to 1$, so the test is inconclusive in this case.

□

4.3 Power Series

One of the most important examples of a series in physics is a **power series**.

Definition (Power series):
A **power series** is a series which is a function of a real variable x of the form;

$$\sum_{k=0}^{\infty} c_k x^k \tag{125}$$

for $c_k \in \mathbb{R}$ for all $k \in \mathbb{N}$.

We state without proof the following theorem:

Theorem 4.1. *Power series*
For a power series there are 3 possibilities.

1. *The series diverges for all $x \in \mathbb{R}$, $x \neq 0$.*
2. *The series converges for all $x \in \mathbb{R}$.*
3. *There exists $r \in \mathbb{R}$ with $r > 0$ such that the series is absolutely convergent for $|x| < r$, diverges for $|x| > r$ and may or may not converge for $x = \pm r$.*

In the latter case r is called the **radius of convergence***.*

Examples:

- $\sum_{k=0}^{\infty} k! x^k$ diverges, by ratio test, for any $x \in \mathbb{R}$.
- $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$ converges, by ratio test, for any $x \in \mathbb{R}$.
- $\sum_{k=0}^{\infty} x^k$ converges for $|x| < 1$, and diverges for $|x| \geq 1$.
- $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$ converges for $|x| < 1$, diverges for $|x| > 1$. It converges for $x = 1$ and diverges for $x = -1$.

Proposition 4.7. *If for a power series $\sum_{k=0}^{\infty} c_k x^k$ the sequence $\left(|c_n|^{\frac{1}{n}}\right)$ converges so $|c_n|^{\frac{1}{n}} \to \frac{1}{r}$ for $r \in \mathbb{R}$, with $r > 0$, then r is the radius of convergence.*

Proof. Consider applying the root test to a power series. Consider the power series $\sum_{k=0}^{\infty} c_k x^k = c_0 + \sum_{n=1}^{\infty} a_n$ with $a_n = c_n x^n$.

Then consider (y_n) with,

$$y_n = |a_n|^{\frac{1}{n}} = |c_n x^n|^{\frac{1}{n}} = |c_n|^{\frac{1}{n}} |x| \tag{126}$$

Suppose the sequence $\left(|c_n|^{\frac{1}{n}}\right)$ converges to a finite real, so $|c_n|^{\frac{1}{n}} \to \frac{1}{r}$ for $r \in \mathbb{R}$, $r > 0$. Then,

$$y_n \to y = \frac{|x|}{r} \quad \text{as} \quad n \to \infty \tag{127}$$

Then by the root test the series is convergent if $y < 1$, ie. $|x| < r$, and is divergent if $y > 1$ ie. $|x| > r$. For $y = 1$, so $|x| = r$ then the series may or may not converge.

$\square$

4.4 Riemann reordering

We have defined series as a sum of a sequence. Now a sequence has a definite ordering of its elements. Does that matter for the series, which after all is simply their sum?

Consider the example;

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) \\
&\quad - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \dots \\
&\quad - \frac{1}{4r} + \left(\frac{1}{2r+1} - \frac{1}{2(2r+1)}\right) - \dots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} + \dots \\
&\quad - \frac{1}{2(2r)} + \frac{1}{2(2r+1)} + \dots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right) \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \qquad (128)
\end{aligned}$$

We have previously shown this series converges, and hence is finite. Thus we have reached a contradiction.

This example illustrates that it **isn't valid to reorder terms in a series**.

Theorem 4.2. *(Riemann reordering)*
Consider the series $\sum_{k=1}^{\infty} a_k$.

- *If it converges absolutely then any reordering of the series converges to the same value.*

- *If it converges, but does* **not** *converge absolutely, then for every real number* $\rho \in \mathbb{R}$ *there exists a reordering of the summands,* (a'_k)*, such that* $\sum_{k=1}^{\infty} a'_k = \rho$.

Proof. (Sketch of proof only)

If a series converges but not absolutely there must be an infinite sequence of positive summands and an infinite sequence of negative summands. Both these sequences must tend to zero.

Let (p_n) be the positive sequence and (q_n) be the negative one. Suppose $\rho > 0$ is the value we want the reordered series to converge to.

Take just enough of the first (p_n)'s so their sum is just larger than ρ.

(Note this is possible as the sum of the (p_n) must diverge, as the series does not converge absolutely).

Now take just enough of the first (q_n)'s so the sum is now just less than ρ.

Now take enough of the next (p_n)'s so the sum is again just larger than ρ.

Again take enough of the next (q_n)'s so the sum is just less than ρ.

Continue in this manner. Since both sequences tend to zero this process converges on ρ.

□