Chapter 7

Homogeneous Equations and Linearly Independent Vectors

In the preceding chapter we have developed methods for solving sets of n linear equations in n unknowns which are of the form $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Cramer's rule was found to provide a formula for the unique solution of such equations if $\det(\mathsf{A}) \neq 0$, but Gaussian elimination is also able to provide solutions in this case, as well as those where $\det(\mathsf{A}) = 0$, where there are either no solutions or infinitely many solutions.

In this chapter, we examine solutions of the associated *homogeneous* equation $\mathsf{A}\boldsymbol{x} = \mathbf{0}$ corresponding to the original *inhomogeneous* equation $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Solutions to the homogeneous equation will be seen to occur naturally when $\det(\mathsf{A}) = 0$. We begin this chapter with examples that illustrate this, which will also provide additional practice in using the method of Gaussian elimination. This will motivate a general expression of solutions of an inhomogeneous equations as the sum of a particular solution of the inhomogeneous equation (a point in $\mathbb{R}^n$) and a solution of the homogeneous equation (a line, plane, or hyperplane in $\mathbb{R}^n$ that contains the particular solution). This leads to the concept of linearly dependent and linearly independent vectors – one of the most important in vector analysis. This will establish a conceptual foundation for understanding the conditions that determine the nature of the solutions of the general linear equation $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ from a more intuitive perspective, as well as providing additional insight into the meaning of the determinant.

An adjunct to linear independence is the notion of a 'natural basis,' which provides the simplest and most convenient set of n vectors in terms of which all other vectors in $\mathbb{R}^n$ can be expressed. These vectors have unit length, are mutually perpendicular and are the generalization to higher dimensions of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ in $\mathbb{R}^3$.

7.1 Introduction

To motivate the representation of solutions of systems of equations $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ when $\det(\mathsf{A}) = 0$ in the form described in the introduction, we consider two examples.

EXAMPLE 7.1. Consider the following system of 3 equations in 3 unknowns:

$$\begin{aligned} x - 2y - 3z &= 2\,, \\ x - 4y - 13z &= 14\,, \\ -3x + 5y + 4z &= 0\,. \end{aligned} \tag{7.1}$$

The determinant of the matrix of coefficients is calculated as

$$\begin{vmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{vmatrix} = -16 - 78 - 15 - (-36) - (-8) - (65) = 0\,, \tag{7.2}$$

so there is no unique solution to the system of equations (7.1). The solution can be obtained by Gaussian elimination. Beginning with

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 1 & -4 & -13 & 14 \\ -3 & 5 & 4 & 0 \end{array}\right), \tag{7.3}$$

we subtract the first row from the second row and add 3 times the first row to the third row:

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & -2 & -10 & 12 \\ 0 & -1 & -5 & 6 \end{array}\right). \tag{7.4}$$

We then subtract twice the third row from each of the first and second rows:

$$\left(\begin{array}{ccc|c} 1 & 0 & 7 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -5 & 6 \end{array}\right). \tag{7.5}$$

Finally, we multiply the third row by -1 and exchange the second and third rows to obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & 7 & -10 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{7.6}$$

The transcription of this matrix into equation form is

$$\begin{aligned} x + 7z &= -10\,, \\ y + 5z &= -6\,. \end{aligned} \tag{7.7}$$

We have two equations in three unknowns, so we do not have a unique solution as we have one degree of freedom. If we set $z = \lambda$, where $\lambda \in \mathbb{R}$, we can write the solution as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 - 7\lambda \\ -6 - 5\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}. \tag{7.8}$$

By defining

$$\boldsymbol{x}_0 \equiv \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x}_1 \equiv \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}, \tag{7.9}$$

we can write (7.8) as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \lambda \boldsymbol{x}_1\,, \tag{7.10}$$

which is the equation of a line (cf. (4.1)). Thus, we have an infinity of solutions that are parametrized by λ.

We now consider the quantities $\mathsf{A}\boldsymbol{x}_0$ and $\mathsf{A}\boldsymbol{x}_1$:

$$\mathsf{A}\boldsymbol{x}_0 = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ 0 \end{pmatrix} = \boldsymbol{b}, \tag{7.11}$$

$$\mathsf{A}\boldsymbol{x}_1 = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}. \tag{7.12}$$

The vector $\boldsymbol{x}_1$ is a solution of the homogeneous equation, and $\boldsymbol{x}_0$ is a **particular solution** of (7.1). Hence, we have

$$\mathsf{A}\boldsymbol{x} = \mathsf{A}(\boldsymbol{x}_0 + \lambda \boldsymbol{x}_1) = \mathsf{A}\boldsymbol{x}_0 + \lambda \mathsf{A}\boldsymbol{x}_1 = \boldsymbol{b}\,. \tag{7.13}$$

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EXAMPLE 7.2. Consider the following system of 3 equations in 3 unknowns:

$$\begin{aligned} 2x+4y-6z &= 6\,, \\ 3x+6y-9z &= 9\,, \\ -x-2y+3z &= -3\,. \end{aligned} \tag{7.14}$$

The determinant of the matrix of coefficients,

$$\begin{vmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{vmatrix} = 36+36+36-36-36-36=0, \tag{7.15}$$

so there is no unique solution to the system of equations (7.14). Gaussian elimination proceeds by first writing

$$\left(\begin{array}{ccc|c} 2 & 4 & -6 & 6 \\ 3 & 6 & -9 & 9 \\ -1 & -2 & 3 & -3 \end{array}\right). \tag{7.16}$$

Adding 2 times row 3 to row 1 and 3 times row 3 to row 2,

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 3 & -3 \end{array}\right), \tag{7.17}$$

multiplying the third row by -1 and exchanging the first and third rows yields,

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{7.18}$$

When transcribed into an equation, Gaussian elimination yields a single constraint among three unknowns:

$$x+2y-3z=3\,, \tag{7.19}$$

which is the equation of a plane (cf. (4.20)). There are two degrees of freedom, which we take as $y=\lambda$ and $z=\mu$ for $\lambda,\mu\in\mathbb{R}$. The solutions to (7.14) can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2\lambda+3\mu+3 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}. \tag{7.20}$$

By defining

$$\boldsymbol{x}_0 \equiv \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x}_1 \equiv \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x}_2 \equiv \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \tag{7.21}$$

we can write (7.20) succinctly as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \lambda x_1 + \mu \boldsymbol{x}_2 . \tag{7.22}$$

Finally, the quantities $\mathsf{A}\boldsymbol{x}_0$, $\mathsf{A}\boldsymbol{x}_1$, $\mathsf{A}\boldsymbol{x}_2$ are calculated as

$$\mathsf{A}\boldsymbol{x}_0 = \begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -1 \end{pmatrix} = \boldsymbol{b}, \tag{7.23}$$

$$\mathsf{A}\boldsymbol{x}_1 = \begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{0}, \tag{7.24}$$

$$\mathsf{A}\boldsymbol{x}_2 = \begin{pmatrix} 2 & 4 & -6 \\ 3 & 6 & -9 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{0}. \tag{7.25}$$

We again see that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are solutions of the homogeneous equation $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$, and $\boldsymbol{x}_0$ is a particular solution of the original system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ in (7.14). ∎

7.2 The Homogeneous Equation

The examples in the preceding section suggest that, for 3×3 systems of equations $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ whose determinant of the matrix of coefficients vanishes, the infinity of solutions can be written as the sum of a particular solution to this equation and solution(s) of the associated homogeneous equation $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$. In this section, we show that this result can be generalized to $n \times n$ systems of equations with infinitely many solutions. We first formalize our discussion with a definition.

DEFINITION 7.1. The n-dimensional zero vector $\boldsymbol{x} = \boldsymbol{0}$ is always a solution of any system of n homogeneous equation in n unknowns: $\mathsf{A}\boldsymbol{0} = \boldsymbol{0}$. This is called the **trivial solution**. Any other solution of these equations is called **non-trivial**.

This definition provides the setting for the following theorem:

THEOREM 7.1. The homogeneous equation $\mathsf{A}\boldsymbol{x} = \mathbf{0}$ has non-trivial solutions if and only if $\det(\mathsf{A}) = 0$.

Before we prove this theorem, we pause to consider its wording. If p and q are statements, then the statement p if and only if q means: If p is true, then q is true, and, conversely, if q is true, then p is true. In other words, statements p and q are both true or both false. This establishes the equivalence of the two statements

Proof. Suppose that $\mathsf{A}\boldsymbol{x} = \mathbf{0}$ has at least one non-trivial solution $\boldsymbol{x} \neq \mathbf{0}$. There is additionally the trivial solution $\boldsymbol{x} = \mathbf{0}$, so this equation has at least two solutions. Hence, the solution is not unique and we conclude from Cramer's rule that $\det(\mathsf{A}) = 0$ and, in fact, that there is an infinity of solutions.

Conversely, suppose the $\det(\mathsf{A}) = 0$. Cramer's rule then implies that there are either no solutions or an infinity of solutions. As $\boldsymbol{x} = 0$ is always a solution, there is at least one solution of the system of equations $\mathsf{A}\boldsymbol{x} = \mathbf{0}$. Hence, there must be an infinite number of non-trivial solutions. $\square$

This theorem establishes the equivalence between a vanishing determinant and non-trivial solutions of the homogeneous equation $\mathsf{A}\boldsymbol{x} = \mathbf{0}$. In particular, consider the solutions of the system of equations $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ where $\det(\mathsf{A}) = 0$ and there is an infinity of solutions. Then, these solutions can be written as

$$\boldsymbol{x} = \boldsymbol{x}_0 + \sum_{i=1}^{k} \lambda \boldsymbol{x}_i\,, \tag{7.26}$$

where $\lambda_i \in \mathbb{R}$, and $\boldsymbol{x}_0$ is a particular solution of the equation and the $\boldsymbol{x}_i$ are solutions of the homogeneous equation: $\mathsf{A}\boldsymbol{x}_0 = \boldsymbol{b}$ and $\mathsf{A}\boldsymbol{x}_i = \mathbf{0}$, for $i = 1, 2, \ldots, k$. We can readily show that (7.26) is a solution of the original equation:

$$\begin{aligned} \mathsf{A}\left(\boldsymbol{x}_0 + \sum_{i=1}^{k} \lambda \boldsymbol{x}_i\right) &= \mathsf{A}\boldsymbol{x}_0 + \mathsf{A}\left(\sum_{i=1}^{k} \lambda \boldsymbol{x}_i\right) = \boldsymbol{b} + \sum_{i=1}^{k} \mathsf{A}(\lambda_i \boldsymbol{x}_i) \\ &= \boldsymbol{b} + \sum_{i-1}^{k} \lambda_i \mathsf{A}\boldsymbol{x}_i = \boldsymbol{b}\,. \end{aligned} \tag{7.27}$$

In arriving at this result, we have used the fact that, for any matrix M, vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, and real numbers λ and μ,

$$\mathsf{M}(\lambda \boldsymbol{u} + \mu \boldsymbol{v}) = \lambda \mathsf{M}\boldsymbol{u} + \mu \mathsf{M}\boldsymbol{v}\,. \tag{7.28}$$

This property is called **linearity** and is one of the most important in mathematical physics, both for matrices and operators generally.

7.3 Linearly Dependent and Independent Vectors

Our discussion of systems of linearly equations has focussed on determining the condition for different types of solutions and finding those solutions either by Cramer's method or, more generally, with Gaussian elimination. In this section, we adopt amore geometric approach to explain how these results can be understood in terms of vectors.

Consider a matrix

$$\mathsf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{2j} & \cdots & a_{nn} \end{pmatrix}. \tag{7.29}$$

We can view the entries in each column as the components of a column vector, just as we did when proving Cramer's rule in Sec. 6.2. Accordingly, we define

$$\boldsymbol{A}_1\equiv\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad \boldsymbol{A}_2\equiv\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{n2} \end{pmatrix}, \quad \boldsymbol{A}_j\equiv\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{nj} \end{pmatrix}, \quad \boldsymbol{A}_n\equiv\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{nn} \end{pmatrix}, \tag{7.30}$$

which enables us to write A as

$$\mathsf{A}=(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots,\boldsymbol{A}_j,\ldots,\boldsymbol{A}_n)\,. \tag{7.31}$$

In particular, a system of n simultaneous homogeneous equations in n unknowns,

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1j} & + & \cdots & + & a_{1n} & = 0, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2j}x_j & + & \cdots & + & a_{2n}x_n & = 0, \\ \vdots & & \vdots & & \ddots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{i1}x_1 & + & a_{i2}x_2 & + & \cdots & + & a_{ij}x_j & + & \cdots & + & a_{in}x_n & = 0, \\ \vdots & & \vdots & & \ddots & & \vdots & & \ddots & & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nj}x_j & + & \cdots & + & a_{nn}x_n & = 0, \end{array} \tag{7.32}$$

can be written concisely as

$$x_1\boldsymbol{A}_1 + x_2\boldsymbol{A}_2 + \cdots + x_j\boldsymbol{A}_j + \cdots + x_n\boldsymbol{A}_n = \boldsymbol{0}. \tag{7.33}$$

If $\det(\mathsf{A}) \neq 0$, then the system of equations $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$ has a unique solution and the corresponding homogeneous system of equations (7.32) has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. Referring to (7.33), this means that the only linear combination of the vectors $\boldsymbol{A}_i$ that equals zero is obtained by setting all coefficients $x_i = 0$.

Alternatively, if $\det(\mathsf{A}) = 0$, then, according Theorem 7.1, the homogeneous equation has non-trivial solutions, which means that there is some linear combination of the $\boldsymbol{A}_i$, with not all x_i equal to zero whose sum is equal to zero. If, say, $x_1 \neq 0$, then we can rearrange (7.33) as

$$\boldsymbol{A}_1 = -\frac{1}{x_1}(x_2\boldsymbol{A}_2 + \cdots + x_j\boldsymbol{A}_j + \cdots + x_n\boldsymbol{A}_n), \tag{7.34}$$

so that $\boldsymbol{A}_1$ can be expressed as a linear combination of the other vectors. This leads us to the following definition:

DEFINITION 7.2. A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the only solution of

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{0}, \tag{7.35}$$

for the coefficients $\{c_1, c_2, \ldots, c_n\}$ is

$$c_1 = c_2 = \cdots = c_n = 0. \tag{7.36}$$

Otherwise, if there is a non-trivial solution of (7.35), the vectors are said to be **linearly dependent**.

The concepts of linear dependence and independence enable us to provide a more intuitive understanding about the nature of solutions to systems of linear equations. An equation of the form $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ can be written as

$$x_1\boldsymbol{A}_1 + x_2\boldsymbol{A}_2 + \cdots + x_j\boldsymbol{A}_j + \cdots + x_n\boldsymbol{A}_n = \boldsymbol{b}. \tag{7.37}$$

In effect, the solution of this system relies on whether there is a linear combination of the $\boldsymbol{A}_i$ which is equal to $\boldsymbol{b}$. If the $\boldsymbol{A}_i$ form a linearly independent set of vectors, then any vector in $\mathbb{R}^n$ can be expressed as a linear combination of these vectors with unique coefficients. In particular, any vector $\boldsymbol{b}$ can be so expressed. This corresponds to the case $\det(\mathsf{A}) \neq 0$ (because the n vectors span $\mathbb{R}^n$) which, according to Cramer's rule, yields a unique solution to the system of equations.

Suppose that the $\boldsymbol{A}_i$ are linearly dependent, so they do not span $\mathbb{R}^n$, though they do span a subspace of $\mathbb{R}^n$. Equation (7.37) can be solved only if $\boldsymbol{b}$ lies with the subspace spanned by the $\boldsymbol{A}_i$, in which case there are infinitely many ways of combining the $\boldsymbol{A}_i$ to obtain $\boldsymbol{b}$, because the subspace has more vectors than needed to span the subspace. If, however, $\boldsymbol{b}$ has a component that lies outside the subspace spanned by the $\boldsymbol{A}_i$, then there is no linear combination of the $\boldsymbol{A}_i$ that produces $\boldsymbol{b}$. Thus, there is no solution to the system of equations. Linearly dependent vectors $\boldsymbol{A}_i$ correspond to the case where $\det(\mathsf{A}) = 0$.

EXAMPLE 7.3. Consider the following system of 2 equations in 2 unknowns:

$$\begin{aligned} x - 2y &= 1\,, \\ 2x - 4y &= 0\,. \end{aligned} \tag{7.38}$$

The vectors $\boldsymbol{A}_1, \boldsymbol{A}_2$, and $\boldsymbol{b}$ are

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.39}$$

from which we see immediately that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are linearly dependent, because $2\boldsymbol{A}_1 + \boldsymbol{A}_2 = \boldsymbol{0}$. These vectors span a one-dimensional subspace of $\mathbb{R}^2$ whose basis can be taken as

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{7.40}$$

This subspace does not include $\boldsymbol{b}$, so there is no solution to this system of equations. We can verify this by Gaussian elimination. we first write

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -4 & 0 \end{array}\right). \tag{7.41}$$

By multiplying the first row by 2 and subtracting from the second row, we obtain

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & -2 \end{array}\right). \tag{7.42}$$

The second equation, $0 = -2$ is clearly false so there is no solution to the system (7.38).

Consider the system of equations

$$\begin{aligned} x - 2y &= 3\,, \\ 2x - 4y &= 6\,, \end{aligned} \tag{7.43}$$

for which

$$\boldsymbol{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 2\boldsymbol{u}. \tag{7.44}$$

Applying Gaussian elimination to the system of equations (7.43), we first have

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -4 & 6 \end{array}\right). \tag{7.45}$$

By again multiplying the first row by 2 and subtracting from the second row, we obtain

$$\left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \end{array}\right), \tag{7.46}$$

which leaves us with a single equation:

$$x - 2y = 3. \tag{7.47}$$

Setting $y = \lambda$. with $\lambda \in \mathbb{R}$ the solution to (7.43) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3+2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{7.48}$$

The particular solution can be understood from the fact that (7.43) can be written as $x\boldsymbol{u} - 2y\boldsymbol{u} = 3\boldsymbol{u}$, or simply as $(x-2y-3)\boldsymbol{u} = \mathbf{0}$, which has a solution $x = 3$, $y = 0$. An equally valid particular solution is $x = 0$ and $y = -\frac{3}{2}$, which is obtained by setting $x = \lambda$ in (7.47) to obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda \\ -\frac{3}{2} + \frac{1}{2}\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{7.49}$$

as a solution of (7.43). ∎

EXAMPLE 7.4. Consider the following system of 3 equations in 2 unknowns:

$$\begin{aligned} x - y &= b_1, \\ 2x + y &= b_2, \end{aligned} \tag{7.50}$$

in which b_1 and b_2 are any real numbers The vectors $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{b}$ are

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{7.51}$$

from which we see, either by inspection, or by calculating their determinant, that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are linearly independent. Hence, any vector $\boldsymbol{b}$ can be written as a unique linear combination of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. The coefficients can be obtained by Gaussian elimination. We first write

$$\left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 2 & 1 & b_2 \end{array}\right). \tag{7.52}$$

We add the first row to the second row and divide the result by 3:

$$\left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 1 & 0 & \frac{1}{3}(b_1+b_2) \end{array}\right). \tag{7.53}$$

Subtracting the second row from the first yields

$$\left(\begin{array}{cc|c} 0 & -1 & \frac{1}{3}(2b_1-b_2) \\ 1 & 0 & \frac{1}{3}(b_1+b_2) \end{array}\right). \tag{7.54}$$

Finally, multiplying the first row by -1 and exchanging the two rows yields the standard form for a unique solution:

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3}(b_1+b_2) \\ 0 & 1 & \frac{1}{3}(b_2-2b_1) \end{array}\right). \tag{7.55}$$

The unique solution to (7.50) is:

$$x = \tfrac{1}{3}(b_1+b_2), \qquad y = \tfrac{1}{3}(b_2-2b_1). \tag{7.56}$$

∎

7.4 Summary

This chapter began with the notion of a homogeneous equation $\mathsf{A}\boldsymbol{x} = \mathbf{0}$. The trivial solution to this equation is $\boldsymbol{x} = \mathbf{0}$, and there are no trivial solutions if and only if $\det(\mathsf{A}) = 0$.

The inhomogeneous equation $\mathsf{A}\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions if and only of the solutions can be written as

$$\boldsymbol{x} = \boldsymbol{x}_9 + \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i\,,$$

where $\mathsf{A}\boldsymbol{x}_0 = \boldsymbol{b}$, $\lambda_i \in \mathbb{R}$, and $\mathsf{A}\boldsymbol{x}_i = \boldsymbol{0}$, for $i = 1, 2, \ldots, k$.

A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ in $\mathbb{R}$ is linearly dependent of there exist numbers $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m = \boldsymbol{0}.$$

A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ in $\mathbb{R}$ is linearly independent if

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m = \boldsymbol{0}$$

implies that $c_1 = c_2 = \cdots = c_m = 0$.