Mathematical Analysis 2017-8

Toby Wiseman

Example sheet 1

Set basics and proof

Question 1.

What is the cardinality of the following sets?

a) $\{0, 1, 2, 3\}$

b) $\emptyset$

c) $\{\emptyset\}$

d) $\{\{5\}\}$

e) $\{\{2, 3, 4\}\}$

f) $\{\mathbb{N}^+, \emptyset\}$

Question 2.

Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1, 3\}$, $D = \{2, 3\}$, $E = \{1\}$, $F = \{2\}$, $G = \{3\}$, $H = \emptyset$. Simplify the following expressions. The answers should be one of $A, B, \ldots, H$.

a) $A \cap B$

b) $A \cup B$

c) $A \cap (B \cap C)$

d) $(C \cup A) \cap B$

e) $A \setminus B$

f) $C \setminus A$

g) $(D \setminus F) \cup (F \setminus D)$

h) $G \setminus A$

i) $A \cup ((B \setminus C) \setminus F)$

j) $H \cup H$

k) $A \cap A$

l) $((B \cup C) \cap C) \cup H$

Question 3.

Consider the sets $A, B, \ldots H$ defined in question 2. Are the following true or false?

a) $\emptyset \in A$

b) $\emptyset \subset A$

c) $2 \in A$

d) $2 \subset A$

e) $\{1\} \in B$

f) $\{1\} \subset B$

Question 4.

Consider the sets $A, B, \ldots H$ defined in question 2.

a) Write out the elements of the set $A \times B$.

b) Write out the elements of the set $C \times C$.

c) What is the cardinality of $A \times H$?

d) Write out the elements of the power set 2^A.

Proof Basics

Question 5.

Prove by contradiction that there are infinitely many natural numbers.

Question 6.

Prove by contradiction;

a) The sum of a rational number and an irrational number is irrational.

b) The product of a non-zero rational number and an irrational number is irrational.

Question 7.

Prove by induction that for $x \neq 1$ and $n \in \mathbb{N}$;

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

Question 8.

Assuming the fundamental theorem of arithmetic, prove that there are infinitely many prime numbers.

Recall the fundamental theorem of arithmetic states: there is a unique prime factorization for any number greater than one.

[Hint: use contradiction and consider a number of the form; $1 + p_1 p_2 p_3 \ldots p_N$, where $p_1, p_2, \ldots p_N$ are prime.]

Question 9.

a) Prove that $\sqrt{2}$ is irrational.

[Hint: Consider for contradiction that $2 = \frac{q^2}{r^2}$, show q and r must have a common factor.]

b) Generalize this proof to show $\sqrt{n}$ is irrational for any prime number n.

[You may use without proof the fundamental theorem of arithmetic.]