Mathematical Analysis 2017-8

Toby Wiseman

Example sheet 1

Set basics and proof

Question 1.

What is the cardinality of the following sets?

a) $\{0, 1, 2, 3\}$

b) $\emptyset$

c) $\{\emptyset\}$

d) $\{\{5\}\}$

e) $\{\{2, 3, 4\}\}$

f) $\{\mathbb{N}^+, \emptyset\}$

Answer:

The cardinalities are;

a) 4 - the set contains 4 things (numbers).

b) 0 - remember $\emptyset$ is an empty set.

c) 1 - $\{\emptyset\}$ is a set containing one thing, the empty set.

d) 1 - $\{\{5\}\}$ contains one thing, the set $\{5\}$.

e) 1 - $\{\{2, 3, 4\}\}$ contains one thing, the set $\{2, 3, 4\}$.

f) 2 - the set contains two things, the set $\mathbb{N}^+$ and the empty set.

Question 2.

Let $A = \{1,2,3\}$, $B = \{1,2\}$, $C = \{1,3\}$, $D = \{2,3\}$, $E = \{1\}$, $F = \{2\}$, $G = \{3\}$, $H = \emptyset$. Simplify the following expressions. The answers should be one of $A, B, \ldots, H$.

a) $A \cap B$

b) $A \cup B$

c) $A \cap (B \cap C)$

d) $(C \cup A) \cap B$

e) $A \setminus B$

f) $C \setminus A$

g) $(D \setminus F) \cup (F \setminus D)$

h) $G \setminus A$

i) $A \cup ((B \setminus C) \setminus F)$

j) $H \cup H$

k) $A \cap A$

l) $((B \cup C) \cap C) \cup H$

Answer:

a) $A \cap B = \{1,2,3\} \cap \{1,2\} = \{1,2\} = B$

b) $A \cup B = \{1,2,3\} \cup \{1,2\} = \{1,2,3\} = A$

c) $A \cap (B \cap C) = \{1,2,3\} \cap (\{1,2\} \cap \{1,3\}) = \{1,2,3\} \cap \{1\} = \{1\} = E$

d) $(C \cup A) \cap B = (\{1,3\} \cup \{1,2,3\}) \cap \{1,2\} = \{1,2,3\} \cap \{1,2\} = \{1,2\} = B$

e) $A \setminus B = \{1,2,3\} \setminus \{1,2\} = \{3\} = G$

f) $C \setminus A = \{1,3\} \setminus \{1,2,3\} = \{\} = H$

g) $(D \setminus F) \cup (F \setminus D) = (\{2,3\} \setminus \{2\}) \cup (\{2\} \setminus \{2,3\}) = \{3\} \cup \{\} = \{3\} = G$

h) $G \setminus A = \{3\} \setminus \{1,2,3\} = \{\} = H$

i) $A \cup ((B \setminus C) \setminus F) = \{1,2,3\} \cup ((\{1,2\} \setminus \{1,3\}) \setminus \{2\}) = \{1,2,3\} \cup (\{2\} \setminus \{2\}) = \{1,2,3\} \cup \{\} = \{1,2,3\} = A$

j) $H \cup H = H$, note that for any set X, then $X \cup X = X$.

k) $A \cap A = A$, likewise for any set X then $X \cap X = X$.

l) $((B \cup C) \cap C) \cup H = ((\{1,2\} \cup \{1,3\}) \cap \{1,3\}) \cup \{\} = (\{1,2,3\} \cap \{1,3\}) \cup \{\} = (\{1,3\}) \cup \{\} = \{1,3\} = C$

Question 3.

Consider the sets $A, B, \ldots H$ defined in question 2. Are the following true or false?

a) $\emptyset \in A$

b) $\emptyset \subset A$

c) $2 \in A$

d) $2 \subset A$

e) $\{1\} \in B$

f) $\{1\} \subset B$

Answer:

a) $\emptyset \in A$ - false. The set A does not contain an element which is the empty set $\{\}$.

b) $\emptyset \subset A$ - true, every set has an empty subset.

c) $2 \in A$ - true

d) $2 \subset A$ - false, since 2 is not a set, it cannot be a subset of anything.

e) $\{1\} \in B$ - false. $B = \{1, 2\}$ and only contains the numbers 1 and 2. The set $\{1\}$ is neither of these.

f) $\{1\} \subset B$ - true. Just taking the element 1 does give a subset of B. Others are $\emptyset$, $\{2\}$ and $\{1, 2\}$.

Question 4.

Consider the sets $A, B, \ldots H$ defined in question 2.

a) Write out the elements of the set $A \times B$.

b) Write out the elements of the set $C \times C$.

c) What is the cardinality of $A \times H$?

d) Write out the elements of the power set 2^A.

Answer:

a) $A \times B = \{\{1,1\},\{2,1\},\{3,1\},\{1,2\},\{2,2\},\{3,2\}\}$

b) $C \times C = \{\{1,1\},\{3,1\},\{1,3\},\{3,3\}\}$

c) A more tricky one; Remember the definition of the cartesian product; $A \times H = \{(a,h)|a \in A\ ,\ h \in H\}$. So, $A \times H = \{(a,h)|a \in \{1,2,3\}\ ,\ h \in \{\}\}$. But there are no $h \in \{\}$, so $A \times H = \emptyset$ as it is empty. The cardinality is zero.

d) The power set 2^A is all subsets of A.

$$2^A = \{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\} \quad (1)$$

Note that $|2^A| = 8 = 2^{|A|}$.

Proof Basics

Question 5.

Prove by contradiction that there are infinitely many natural numbers.

Answer:

Proposition: There are infinitely many natural numbers.

Proof. Suppose for contradiction that the set of natural numbers, $\mathbb{N}$, is finite.
Then there exists a greatest natural number N ie. $n < N$ for all other $n \in \mathbb{N}$.
However $(1 + N) \in \mathbb{N}$ and $(1 + N) > N$ which is a contradiction.
Hence the proposition is true.

$\square$

Question 6.

Prove by contradiction;

a) The sum of a rational number and an irrational number is irrational.

b) The product of a non-zero rational number and an irrational number is irrational.

Answer:

Proposition: If $q \in \mathbb{Q}$ and $r \in (\mathbb{R} \setminus \mathbb{Q})$ then $q + r \in (\mathbb{R} \setminus \mathbb{Q})$.

Proof. Since q is rational we may write $q = \frac{n}{m}$ for some $n \in \mathbb{Z}$, $m \in \mathbb{N}^+$.

Assume for contradiction that $q + r$ is also rational so we may write $(q + r) = \frac{N}{M}$ for some $N \in \mathbb{Z}$, $M \in \mathbb{N}^+$.

Then,

$$r = (q + r) - q = \frac{N}{M} - \frac{n}{m} = \frac{Nm - nM}{Mm} = \frac{n'}{m'} \tag{2}$$

However we see $r = \frac{n'}{m'}$ is a ratio of $n' = Nm - nM \in \mathbb{Z}$, $m' = Mm \in \mathbb{N}^+$ which is a contradiction since r is irrational.

Hence the proposition is true.

□

Proposition: If $q \in \mathbb{Q} \setminus \{0\}$ and $r \in (\mathbb{R} \setminus \mathbb{Q})$ then $q \cdot r \in (\mathbb{R} \setminus \mathbb{Q})$.

Proof. Since q is rational and non-zero we may write $q = \frac{n}{m}$ for some $n \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{N}^+$.

Assume for contradiction that $q \cdot r$ is also rational so we may write $q \cdot r = \frac{N}{M}$ for some $N \in \mathbb{Z}$, $M \in \mathbb{N}^+$.

Then since $q \neq 0$,

$$r = \frac{q \cdot r}{q} = \frac{\frac{N}{M}}{\frac{n}{m}} = \frac{n'}{m'} \tag{3}$$

However we see $r = \frac{n'}{m'}$ is a ratio of $n' = Nm \in \mathbb{Z}$, $m' = nM \in \mathbb{Z} \setminus \{0\}$, and so is rational, which is a contradiction since r is irrational.

Hence the proposition is true.

□

Question 7.

Prove by induction that for $x \neq 1$ and $n \in \mathbb{N}$;

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

Answer:

Proposition: $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ for $x \neq 1$ and $n \in \mathbb{N}$.

Proof. We use induction;

- For $n = 0$ the lhs $= x^0 = 1$ and the rhs $= \frac{1-x^1}{1-x} = 1$ so the claim is true.
- Suppose the claim is true for some $n \in \mathbb{N}$. Then (for $x \neq 1$),

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + \frac{(1 - x)x^{n+1}}{1 - x}$$
$$= \frac{1 - x^{n+1}}{1 - x} + \frac{x^{n+1} - x^{n+2}}{1 - x} = \frac{1 - x^{n+2}}{1 - x}$$

  showing the claim also holds for $(n + 1)$.

By induction the claim is true for all $n \in \mathbb{N}$.

□

Question 8.

Assuming the fundamental theorem of arithmetic, prove that there are infinitely many prime numbers.

Recall the fundamental theorem of arithmetic states: there is a unique prime factorization for any number greater than one.

[Hint: use contradiction and consider a number of the form; $1 + p_1 p_2 p_3 \ldots p_N$, where $p_1, p_2, \ldots p_N$ are prime.]

Answer:

Proposition: there are infinitely many prime numbers.

Proof. Assume for contradiction there are finitely many primes, say N. We may list them in order,

$$p_1 < p_2 < p_3 < \ldots < p_{N-1} < p_N \tag{4}$$

with p_N being the largest prime. Now consider the number which is the product of all the primes plus one, so,

$$Z = 1 + p_1 p_2 p_3 \ldots p_{N-1} p_N \tag{5}$$

However Z divided by any of the primes p_i, $1 \leq i \leq N$ has remainder one. Hence $Z > 1$ has no prime factors, and therefore (by the fundamental thm of arithmetic) must be prime itself. But since $Z > p_N$ this is a contradiction. Therefore there are infinitely many primes. $\square$

Question 9.

a) Prove that $\sqrt{2}$ is irrational.

[Hint: Consider for contradiction that $2 = \frac{q^2}{r^2}$, show q and r must have a common factor.]

b) Generalize this proof to show $\sqrt{n}$ is irrational for any prime number n.

[You may use without proof the fundamental theorem of arithmetic.]

Answer:

Proposition: $\sqrt{2}$ is not rational.

Proof. Assume for contradiction that $\sqrt{2}$ is rational.Then there exists $p, q \in \mathbb{N}^+$ such that,

$$\sqrt{2} = \frac{p}{q} \tag{6}$$

where p, q have no common factors. Then squaring,

$$2 = \frac{p^2}{q^2} \tag{7}$$

implies p^2 is even and hence since p is a natural number, p is even. Then we may write $p = 2r$ for some $r \in \mathbb{N}^+$. Now we have,

$$2 = \frac{4r^2}{q^2} \implies 2 = \frac{q^2}{r^2} \tag{8}$$

which now implies q^2 is even and hence q is even. But if p and q are both even then they have the common factor two which is a contradiction. Hence $\sqrt{2}$ is not rational ie. is irrational. □

The key to this proof was seeing that if $p \in \mathbb{N}^+$ and p^2 is even, then so is p. More generally the following holds;

Proposition: Let $p \in \mathbb{N}^+$. If p^2 has prime factor n, so does p.

Proof. By the fundamental theorem of arithmetic we may uniquely write $p = p_1^{n_1} p_2^{n_2} p_3^{n_3} \dots p_k^{n_k}$ for some $k \in \mathbb{N}^+$, and $n_i \in \mathbb{N}^+$. Then,

$$p^2 = p_1^{2n_1} p_2^{2n_2} p_3^{2n_3} \dots p_k^{2n_k} \tag{9}$$

Thus if p^2 has a prime factor n, it has it at least twice, and p also has it at least once.

□

Proposition: $\sqrt{n}$ is not rational for n prime.

Proof. Assume for contradiction that $\sqrt{n}$ is rational.Then there exists $p, q \in \mathbb{N}^+$ such that,

$$\sqrt{n} = \frac{p}{q} \tag{10}$$

where p, q have no common factors. Then squaring,

$$n = \frac{p^2}{q^2} \tag{11}$$

implies $p^2 = nq^2$. Thus the number p^2 has a prime factor n. Hence (by our proposition above) the number p must also have a prime factor n. Thus we may write, $p = nr$ for some $r \in \mathbb{N}^+$. Now we have,

$$n = \frac{n^2 r^2}{q^2} \quad \Longrightarrow \quad n = \frac{q^2}{r^2} \tag{12}$$

which now implies q^2, and hence q also have prime factor n. But if p and q both share the common prime factor n this is a contradiction. Hence $\sqrt{n}$ is not rational ie. is irrational. □