Mathematical Analysis 2017-8

Toby Wiseman

Example sheet 2

Set and maps

Question 1.

The following 'definitions' do not define proper maps. In each case explain why. In each case change the domain and/or target sets to make a proper map (note: there isn't a single correct way to do this).

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \arcsin x$

b) $f : \mathbb{N}^+ \to \mathbb{N}^+$, $f(x) = \ln x$

c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \ln x$

d) $f : \mathbb{Z} \to \{\emptyset, \{\emptyset\}\}$, $f(x) = \begin{cases} \emptyset & x > 0 \\ \{\emptyset\} & x < 0 \end{cases}$

e) $f : \mathbb{N}^+ \to \emptyset$, $f(x) = \emptyset$

f) For a finite set A; $f : 2^A \to \mathbb{N}^+$, $f(x) = |x|$

g) For a set A and non-empty finite set B; $f : 2^A \to \mathbb{N}^+$, $f(x) = |x \cup B|$

Answer:

a) There are points in the domain for which the map is not defined; for example arcsin(2) is not defined, since $\sin x \leq 1$ for all $x \in \mathbb{R}$. We must change the domain to $[-1, 1]$.

We must also be careful; for $x \in [-1, 1]$ there is no unique $y \in \mathbb{R}$ such that $\sin y = x$. (If this is true for y, it is true for $y + 2\pi n$ for any $n \in \mathbb{N}$).

However using the usual convention that $y = \arcsin x$ returns $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ such that $\sin y = x$, then the target is fine.

Hence we can define the map;

$$f : [-1, 1] \to \mathbb{R} \ , \ f(x) = \arcsin x \tag{1}$$

or we could define,

$$f : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \ , \ f(x) = \arcsin x \tag{2}$$

Either is fine (the latter being minimal given the domain).

b) There are points in the domain that do not get mapped to those in the target; for example $\ln(2) = 0.693\ldots \notin \mathbb{N}^+$. Note that $\ln(n) \geq 0$ for all $n \in \mathbb{N}^+$. So we could take,

$$f : \mathbb{N}^+ \to [0, \infty) \ , \ f(x) = \ln x \tag{3}$$

where $[0, \infty) = \{x \in \mathbb{R} | x \geq 0\}$, or simply,

$$f : \mathbb{N}^+ \to \mathbb{R} \ , \ f(x) = \ln x \tag{4}$$

c) This map is not defined as for $x = 0$, $\ln(x)$ is not defined, and for $x < 0$, $\ln(x) \notin \mathbb{R}$, but is complex. Hence we could just take,

$$f : (0, \infty) \to \mathbb{R} \ , \ f(x) = \ln x \tag{5}$$

where $(0, \infty) = \{x \in \mathbb{R} | x > 0\}$. Or we could just remove 0 from the domain make make the target complex,

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{C} \ , \ f(x) = \ln x \tag{6}$$

d) This map is not defined for the domain element 0. Thus we remove that element from the domain,

$$f : \mathbb{Z} \setminus \{0\} \to \{\emptyset, \{\emptyset\}\} \, , \ f(x) = \begin{cases} \emptyset & x > 0 \\ \{\emptyset\} & x < 0 \end{cases} \tag{7}$$

e) This map is bad as the function returns a value not in the target. For any $x \in \mathbb{N}^+$ in the domain, the function returns $\emptyset$, ie. a set that is empty. However the target is the empty set, so there are no elements in the target, in particular there is no element which is a set (empty or otherwise). We can fix this by amending the target to include the empty set;

$$f : \mathbb{N}^+ \to \{\emptyset\} \, , \ f(x) = \emptyset \tag{8}$$

f) This map is bad as there is an element in the domain which is not mapped to the target. The empty set is always a subset of any set (so $\emptyset \in 2^A$), and since $|\emptyset| = 0$, we must include 0 in the target. So,

$$f : 2^A \to \mathbb{N} \, , \ f(x) = |x| \tag{9}$$

for a finite set A is fine.

g) This map is bad as there are elements in the domain which are not well defined. If A is a finite set, the map is fine. However if A is infinite, there will be elements of the domain, 2^A, which are themselves infinite sets - possibly not even countable. Suppose x is one, then so is $x \cup B$ and so $|x \cup B|$ is not defined as a natural number.

As in the previous question we may have a reasonable map by restricting the domain to finite sets. Note that since B is not empty, then $x \cup B$ is not empty, so the target is fine to be $\mathbb{N}^+$ rather than needing $\mathbb{N}$. So;

For a **finite** set A and a finite set B; $f : 2^A \to \mathbb{N}^+$, $f(x) = |x \cup B|$

Question 2.

State if the following maps are injective or not, and whether they are surjective or not. Explain your answers.

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$

b) $f : \mathbb{R} \to \{x \in \mathbb{R} | x > 0\}$, $f(x) = e^{-x^2}$

c) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$

d) $f : \mathbb{N}^+ \to \{\emptyset, \{\emptyset\}\}$, $f(x) = \begin{cases} \emptyset & x \text{ even} \\ \{\emptyset\} & x \text{ odd} \end{cases}$

Answer:

a) This map is injective since $e^x = e^y$ only if $x = y$.

The map is not surjective as $e^x > 0$ for all $x \in \mathbb{R}$. For example, there is no element in the domain that maps to the element -1 in the target.

b) This map is not injective since $e^{-x^2} = e^{-y^2}$ if $x = y$ or if $x = -y$.

The map is not surjective as there is not $x \in \mathbb{R}$ such that $e^{-x^2} > 1$.

c) This map is injective since for $x \neq 0$ then $\frac{1}{x} = \frac{1}{y}$ only if $x = y$.

The map is not surjective however, since the element 0 in the target is not the image of any element in the domain.

d) This map is not injective since any even positive natural number is mapped to $\emptyset$.

The map is surjective; both elements of the target are the image of elements in the domain, for example $f(1) = \{\emptyset\}$ and $f(2) = \emptyset$.

Question 3.

Suppose 98 students are following the Maths Analysis course and all wish to come to office hour on Monday 12-1. I have 3 chairs in my office that can accommodate only one person. Use the pigeon hole principle to prove that not everyone who comes to office hour will be able to sit down on a chair. Be careful to identify properly the sets and maps involved.

Answer:

Let the set of students be S. Then $|S| = 98$.

Let the set of chairs in my office be C. So $|C| = 3$.

Consider the office hour map, f, of students to chairs; $f : S \to C$.

By the pigeon hole principle since $|S| > |C|$ then f cannot be an injection. Hence there must always exist some students $x, y \in S$ such that $x \neq y$ but $f(x) = f(y)$, so they are mapped to the same chair.

Thus not every student can have their own chair (since a chair can only accommodate one student).

Question 4.

Prove the generalized pigeon hole principle below;

Theorem: Consider a function $f : D \to T$, where D and T are finite sets. Let $N = |D|$ and $M = |T|$, and let $N > M$. Then there is some element of T such that f maps at least $\lceil \frac{N}{M} \rceil$ elements of D to it.

Alternatively; Given N pigeons and M pigeon holes, and $N > M$, then one pigeon hole will have at least $\lceil \frac{N}{M} \rceil$ pigeons in it.

Here we use the notation that for $x \in \mathbb{R}$ then $\lceil x \rceil$ is the smallest integer which is greater or equal to x.

Hint: you may find a contradiction argument useful.

Answer:

Proof. Assume the opposite for contradiction, that every element of the target ('pigeon hole') has less than or equal to $\left(\lceil \frac{N}{M} \rceil - 1\right)$ elements in the domain mapped to it ('pigeons in it').

Then the total number of elements in the domain ('pigeons'), N, is than less or equal to $M \times \left(\lceil \frac{N}{M} \rceil - 1\right)$.

But, $M \times \lceil \frac{N}{M} \rceil < M \times \left(\frac{N}{M} + 1\right) = N + M$ so we see $M \times \left(\lceil \frac{N}{M} \rceil - 1\right) < N$.

This is a contradiction. Hence our assumption must be false.

□

Question 5.

Consider the mathematical analysis exam, which will have the same format as the previous year, so students must choose 3 questions from 5. Suppose 98 students sit the exam. Prove that at least 10 students will choose to answer exactly the same set of questions.

Hint: use the generalized pigeon hole principle.

Answer:

Let the set of students be S, so that $|S| = 98$.

Let set of possible sets of questions answered be Q. Then $|Q| = \left(\begin{array}{c}5\\3\end{array}\right) = \frac{5!}{3!(5-3)!} = 10.$

Consider the map $f : S \to Q$ that describes the set of questions that each student chooses. Then by the generalied pigeon hole principle, since $|S| > |Q|$, one element of the target must have at least $\lceil \frac{98}{10} \rceil = \lceil 9.8 \rceil = 10$ elements mapped to it - ie. one set of questions must have at least at least 10 students choose it.