Mathematical Analysis 2017-8

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Example sheet 3

Numbers, counting and infinity

Question 1.

What is the cardinality of the following sets;

a) $\emptyset$

b) $\{\emptyset\}$

c) $\{\emptyset, \{\emptyset\}\}$

d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

[Such sets can be used to define the natural numbers 'from nothing'.]

Answer:

a) $|\emptyset| = 0$ as $\emptyset = \{\}$ contains no elements.

b) $|\{\emptyset\}| = 1$. There is one element in this set (the element is the empty set).

c) $|\{\emptyset, \{\emptyset\}\}| = 2$. There are two elements, the sets $\emptyset$ and $\{\emptyset\}$.

d) $|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| = 3$. There are 3 elements, the sets $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$.

Question 2.

Find a rational expression for the following periodic decimals;

a) $2.2222222\ldots$ (ie. $2.\dot{2}$)

b) $0.84090909\ldots$ (ie. $0.84\dot{0}\dot{9}$)

c) $1.542303303303\ldots$ (ie. $1.542\dot{3}0\dot{3}$)

You may use the following result for a geometric series; for $x \in \mathbb{R}$ and $|x| < 1$ then,

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

Answer:

a)

$$\begin{aligned} 2.2222222\ldots &= 2 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \ldots \\ &= 2\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\right) \\ &= 2 \cdot \frac{1}{1 - \frac{1}{10}} = 2 \cdot \frac{10}{9} = \frac{20}{9} \end{aligned} \tag{1}$$

b)

$$\begin{aligned} 0.84090909\ldots &= 0.84 + \frac{9}{10000} + \frac{9}{1000000} + \frac{9}{100000000} + \ldots \\ &= \frac{84}{100} + \frac{9}{10000}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \ldots\right) \\ &= \frac{21}{25} + \frac{9}{10000} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{84}{100} + \frac{9}{10000}\frac{100}{99} \\ &= \frac{21}{25} + \frac{1}{1100} = \frac{44 \times 21 + 1}{1100} = \frac{925}{1100} = \frac{37}{44} \end{aligned} \tag{2}$$

c)

$$
\begin{aligned}
1.542303303303\ldots &= 1.542 + \frac{303}{1000000}\left(1 + \frac{1}{1000} + \frac{1}{1000^2} + \frac{1}{1000^3} + + \ldots\right) \\
&= \frac{1542}{1000} + \frac{303}{1000000} \cdot \frac{1}{1 - \frac{1}{1000}} = \frac{771}{500} + \frac{303}{1000000}\frac{1000}{999} \\
&= \frac{771}{500} + \frac{303}{999000} = \frac{771 \times 1998 + 303}{999000} \\
&= \frac{1540761}{999000} \\
&= \frac{513587}{333000} \qquad (3)
\end{aligned}
$$

Question 3.

In lectures we will prove $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. Assuming this result, prove $|\mathbb{R}^n| = |\mathbb{R}|$ for any $n \in \mathbb{N}^+$.

Answer:

Claim: $|\mathbb{R}^n| = |\mathbb{R}|$ for $n \in \mathbb{N}^+$.

Proof. We use induction;

- The proposition is obviously true for $n = 1$, and by assuming $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ also for $n = 2$.
- Suppose the proposition is true for some $n \in \mathbb{N}^+$, $n > 2$. Then $|\mathbb{R}^n| = |\mathbb{R}|$ for that n and so there exists a bijection $f : \mathbb{R}^n \to \mathbb{R}$.

  Now consider the map built from this;

  $$g : \mathbb{R}^{n+1} \to \mathbb{R} \times \mathbb{R}$$
  $$x = (a_1, a_2, \ldots, a_n, b) \to g(x) = (f(a_1, a_2, \ldots, a_n), b)$$

  This is a bijection (since f is a bijection).

  Since we assume $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ there exists a bijection $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

  Then the map $h \circ g$ is a bijection, $h \circ g : \mathbb{R}^{n+1} \to \mathbb{R}$. Hence the proposition also holds for $(n + 1)$ if it holds for n.

By induction the proposition holds for all $n \in \mathbb{N}^+$.

□

Question 4.

Prove that if F is a finite set, and I is a countable set, then $I \cup F$ is countable.

[Hint: in lectures we proved $I_1 \cup I_2$ was countable if $I_{1,2}$ were countable sets - try a similar argument.]

Answer:

Claim: If F is a finite set and I is a countable set, then $I \cup F$ is countable.

Proof. Suppose $|F| = n$. We may list the elements of $F = \{f_1, f_2, \ldots f_n\}$.

Since I is countable we may list its elements, $I = \{i_1, i_2, i_3, \ldots\}$.

Then we may list the elements of $I \cup F$ as $\{f_1, f_2, \ldots f_n, i_1, i_2, i_3, \ldots\}$ where we omit any repetitions in the listing (at most n of them).

Hence $I \cup F$ is infinite but listable so it is countable.

□

Question 5.

Let both $f : A \to B$ and $g : X \to Y$ be bijections. Let $A \cap X = \emptyset$ and $B \cap Y = \emptyset$. Prove that,

$$\begin{aligned} h: \quad & A \cup X \to B \cup Y \\ & x \to h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in X \end{cases} \end{aligned}$$

is also a bijection.

Answer:

Claim: The map h (defined above) is a bijection.

Proof. Firstly we show h is an injection. Consider any $x, y \in A \cup X$. Since $A \cap X = \emptyset$ are 4 cases;

- if $x, y \in A$ then $x \neq y$ implies $h(x) \neq h(y)$ since f is an injection.
- if $x, y \in X$, then $x \neq y$ implies $h(x) \neq h(y)$ since g is an injection.
- if $x \in A$ and $y \in X$ then $x \neq y$ since $A \cap X = \emptyset$ and $h(x) \neq h(y)$ since $f(x) \neq g(y)$ as $B \cap Y = \emptyset$.
- likewise for $x \in X$ and $y \in A$ we have $x \neq y$ and $h(x) \neq h(y)$.

Thus if $x \neq y$ then we have $h(x) \neq h(y)$ and hence h is an injection.

Second we show it is a surjection. Consider $y \in B \cup Y$. Since $B \cap Y = \emptyset$ there are 2 cases;

- if $y \in B$ then $x = f^{-1}(y) \in A \subset A \cup X$ so that $h(x) = y$.
- if $y \in Y$ then $x = g^{-1}(y) \in X \subset A \cup X$ so that $h(x) = y$.

Hence for any $y \in B \cup Y$ there exists $x \in A \cup X$ such that $h(x) = y$. Hence h is surjective. $\square$

Question 6.

Suppose C is a countable set and I is an infinite (not necessarily countable) set such that $C \cap I = \emptyset$. Prove that $|C \cup I| = |I|$.

[Hint: I has a countable subset - call this K - and so we can decompose I as $I = (I \setminus K) \cup K$. Use this and the result in question 5 to construct a bijection between $C \cup I$ and I.]

Answer:

Claim: Let C and I be infinite sets such that C is countable and $C \cap I = \emptyset$. Then $|C \cup I| = |I|$.

Proof. I is an infinite set and thus has a countable subset K so $I = K \cup (I \setminus K)$.

We will construct a bijection using the result from question 5.

Let $A = K$ and $X = (I \setminus K)$ so that $A \cup X = I$. Note that $A \cap X = \emptyset$.

Let $B = (C \cup K)$ and $Y = (I \setminus K)$ so that $B \cup Y = C \cup I$. Note $B \cap Y = \emptyset$.

Now A and B are countable so there exists a bijection $f : A \to B$.

Since $X = Y$ there exists the trivial bijection $id_X : X \to Y$.

From question 5 the following is then a bijection from I to $C \cup I$;

$$h : \quad A \cup X \to B \cup Y$$
$$x \to h(x) = \begin{cases} f(x) & x \in A \\ x & x \in X \end{cases}$$

Hence $|I| = |C \cup I|$.

□

Question 7.

Consider the Cartesian product of countably infinitely many copies of the set $\{0, 1\}$;

$$K = \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \dots$$

Elements of K are then infinite sequences of zeros and ones eg. $(0, 1, 1, 0, 0, 0, 1, 1, \dots) \in K$.

Prove that $|K| = |\mathbb{R}|$.

[Hint: Use the binary (ie. base 2) number system to represent a real number. Just as for decimals where we exclude numbers ending in recurring 9's, be careful with the binary numbers. You will need the result from question 6.]

Answer:

Proof. Use binary representation of real numbers eg.

$$\begin{aligned} 11.011\dot{0} &= 2 + 1 + 0 + \frac{1}{4} + \frac{1}{8} \\ &= 3.375\dot{0} \end{aligned} \tag{4}$$

Recall for decimals we excluded the representation ending in recurring 9's (as it is redundant). Likewise for binary we exclude the form ending in recurring 1's.

Now partition K into 3 non-intersecting subsets; $K = K_1 \cup K_2 \cup K_3$, where,

- $K_1 = \{(0, 0, 0, 0, 0, \dots)\}$ ie. the set with one element which only contains zeros.
- K_2 =set of all sequences that end in recurring ones.
- $K_3 = K \setminus (K_1 \cup K_2)$ ie. all the rest!

Firstly $|K_1| = 1$.

Secondly the set K_2 is countable as we can list the elements,

$$\begin{aligned} K_2 = \{ & \\ & (1, 1, 1, 1, \dots), \\ & (0, 1, 1, 1, \dots), \\ & (1, 0, 1, 1, \dots), \\ & (0, 0, 1, 1, \dots), \\ & (1, 1, 0, 1, \dots), \\ \dots \} & \end{aligned} \tag{5}$$

Thirdly, using binary infinite decimals we have a bijection,

$$\begin{aligned} f: \quad K_3 &\to (0,1) \\ (a_1, a_2, a_3, \ldots) &\to 0.a_1a_2a_3\ldots \end{aligned} \tag{6}$$

so $|K_3| = |(0,1)| = |\mathbb{R}|$.

Then $K = K_3 \cup (K_1 \cup K_2)$ and $K_1 \cup K_2$ is countable, with $K_3 \cap (K_1 \cup K_2) = \emptyset$ so by the result in question 6 we have $|K| = |K_3| = |R|$.

$\square$

Harder questions: if you have time...

Question 8.

Prove that $\left|2^{\mathbb{N}^+}\right| = |\mathbb{R}|$ by finding an injection from $\mathbb{R} \to 2^{\mathbb{N}^+}$ and another from $2^{\mathbb{N}^+} \to \mathbb{R}$.

[Hint: use infinite decimals to construct these injections. Make sure the maps you construct really are injections.]

Answer:

Claim: $|(0,1)| \leq \left|2^{\mathbb{N}^+}\right|$ ie. there exists an injection $f : (0,1) \to 2^{\mathbb{N}^+}$.

Proof. We construct an injection explicitly using infinite decimals (there are many ways to do this). Consider the map;

$$\begin{aligned} f : \qquad (0,1) &\to 2^{\mathbb{N}^+} \\ x = 0.a_1a_2a_3\ldots &\to \{n_1, n_2, n_3, \ldots\} \end{aligned}$$

where we use infinite decimal form on the l.h.s. above, and,

$$\begin{aligned} n_1 &= 1a_1 \\ n_2 &= 1a_2a_3 \\ n_3 &= 1a_4a_5a_6 \\ n_4 &= 1a_7a_8a_9a_{10} \\ &\vdots \end{aligned}$$

using usual decimal form for the natural numbers n_i.

Note that adding the leading one's above ensures the map is an injection, even if some of the digits a_i are zero's. The numbers $n_i \neq n_j$ unless $i = j$ as they have different numbers of digits. Then every real is uniquely mapped to a set of positive natural numbers.

□

For example;

$x = 0.2854368652\ldots \to f(x) = \{12, 185, 1436, 18652, \ldots\}$.

$x = 0.285\dot{0} \to f(x) = \{12, 185, 1000, 10000, \ldots\}$.

Claim: $\left|2^{\mathbb{N}^+}\right| \leq |(0,1)|$ ie. there exists an injection $g : 2^{\mathbb{N}^+} \to (0,1)$.

Proof. Consider the map;

$$\begin{aligned} g : \qquad 2^{\mathbb{N}^+} &\to (0,1) \\ x = \{n_1, n_2, n_3, \ldots\} &\to g(x) = 0.a_1 a_2 a_3 \ldots \end{aligned}$$

where we use infinite decimal form on the r.h.s. above and take,

$$a_k = \begin{cases} 2 & k \in x \\ 1 & k \notin x \end{cases} \qquad \forall\, k \in \mathbb{N}^+$$

This is an injection - every element of $2^{\mathbb{N}^+}$ is uniquely mapped to a real in $(0,1)$.

□

For example;

$x = \{4, 2, 10\} \to f(x) = 0.12121111112\dot{1}$.

$x = \{2, 4, 10, 254, 2532, \ldots\} \to f(x) = 0.121211111211111\ldots$.

Claim: $|\mathbb{R}| = \left|2^{\mathbb{N}^+}\right|$

Proof. Consider the bijection $h : (0,1) \to \mathbb{R}$ given by $h(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$.

Then $f \circ h^{-1} : \mathbb{R} \to 2^{\mathbb{N}^+}$ is an injection.

Then $h \circ g : 2^{\mathbb{N}^+} \to \mathbb{R}$ is an injection.

Hence by Cantor-Berstein-Schroder there exists a bijection $k : \mathbb{R} \to 2^{\mathbb{N}^+}$. Hence $|\mathbb{R}| = \left|2^{\mathbb{N}^+}\right|$.

□