Mathematical Analysis 2017-8

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Example sheet 4

Sequences

Question 1.

Let $r \in \mathbb{R}$ and x_n be its infinite decimal truncated to n decimal places. eg. for $r = \pi$ then $x_4 = 3.1415$.

a) Prove that the sequence (x_n) is a Cauchy sequence.

b) Prove that $x_n \to r$ as $n \to \infty$.

Answer:

Claim: (x_n) is a Cauchy sequence.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}^+$ such that $\frac{1}{10^N} < \epsilon$.

Consider $m > n > N$. Then as infinite decimals,

$$x_n = a.d_1 d_2 \dots d_n \dot{0}$$
$$x_m = a.d_1 d_2 \dots d_n d_{n+1} \dots d_m \dot{0}$$

so,

$$|x_m - x_n| = 0.0 \dots 0 d_{n+1} d_{n+2} \dots d_m \dot{0} < \frac{1}{10^n} < \frac{1}{10^N} < \epsilon$$

Hence $|x_m - x_n| < \epsilon$ for all $m, n > N$.

□

Claim: $x_n \to r$ as $n \to \infty$.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}^+$ such that $\frac{1}{10^N} < \epsilon$.

Consider $n > N$. Then as infinite decimals,

$$r = a.d_1 d_2 \dots d_n d_{n+1} \dots$$
$$x_n = a.d_1 d_2 \dots d_n \dot{0}$$

So we have,

$$|x_n - r| = r - x_n = 0.0\ldots0d_{n+1}d_{n+2}\ldots < \frac{1}{10^n} < \frac{1}{10^N} < \epsilon$$

□

Question 2.

Prove using $\epsilon - N$ the following;

1. Let $a_n = \frac{1}{\sqrt{n}}$. Then $a_n \to 0$ as $n \to \infty$.
2. Let $b_n \to b$ be a convergent sequence. Let $x_n = b_n^2$. Then $x_n \to b^2$ as $n \to \infty$.
3. Let $c_n \to c$ and $d_n \to d$ be convergent sequences. Let $x_n = c_n - d_n$. Then $x_n \to c - d$ as $n \to \infty$.

Answer:

Claim: The sequence $a_n = \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.

Proof. Let $\epsilon > 0$. Then choose $N \in \mathbb{N}^+$ such that $N > \frac{1}{\epsilon^2}$. Then for $n > N$ we have $n > \frac{1}{\epsilon^2}$ so,

$$|a_n - 0| = \frac{1}{\sqrt{n}} < \epsilon$$

□

Claim: If $b_n \to b$ as $n \to \infty$ then $x_n = b_n^2 \to b^2$.

Rough work:

$$b_n^2 - b^2 = (b_n - b)^2 + 2b(b_n - b)$$

End of rough work

Proof. Let $\epsilon > 0$. Since $b_n \to b$ we may choose $N \in \mathbb{N}^+$ such that for $n > N$ both $|b_n - b| < \sqrt{\frac{\epsilon}{2}}$ and $|b_n - b| < \frac{\epsilon}{4|b|}$. Then for $n > N$,

$$\left|x_n - b^2\right| = \left|(b_n - b)^2 + 2b(b_n - b)\right| \leq |b_n - b|^2 + 2|b|\,|b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Claim: If $c_n \to c$ and $d_n \to d$ as $n \to \infty$ then $x_n = c_n - d_n \to c - d$.

Proof. Let $\epsilon > 0$. Since $c_n \to c$ and $d_n \to d$ as $n \to \infty$ then we may find an $N \in \mathbb{N}^+$ such that for $n > N$ then both $|c_n - c| < \frac{\epsilon}{2}$ and $|d_n - d| < \frac{\epsilon}{2}$. Then for $n > N$ we have,

$$|x_n - (c - d)| = |(c_n - c) - (d_n - d)| \leq |c_n - c| + |d_n - d| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Question 3.

Consider a sequence (y_n) where each term is non-zero, $y_n \neq 0$, with a non-zero limit $y_n \to y$ as $n \to \infty$ (so $y \neq 0$). Prove using $\epsilon - N$ that the sequence (u_n) defined by $u_n = \frac{1}{y_n}$ converges to $u_n \to \frac{1}{y}$ as $n \to \infty$.

Answer:

Rough work:

$$\frac{1}{y_n} - \frac{1}{y} = \frac{y - y_n}{y y_n} \tag{1}$$

so that,

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y - y_n|}{|y|\,|y_n|} \tag{2}$$

Now since $y_n \to y$, then for suitably large n we have $|y_n| > a$ for any $a < y$. *End of rough work*

Proof. Let $\epsilon > 0$. Choose any a such that $0 < a < |y|$ (for example we could choose $a = |y|/2$).

Since $y_n \to y \neq 0$, there exists $N \in \mathbb{N}^+$ such that for $n > N$ then,

$$|y_n - y| < \epsilon |y| a \quad \text{and} \quad |y_n| > a \tag{3}$$

Then for $n > N$ we have,

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y - y_n|}{|y|\,|y_n|} < \frac{(\epsilon |y| a)}{|y| a} = \epsilon \tag{4}$$

□

Question 4.

Consider a sequence (x_n) with a limit $x_n \to x$ where $x > 0$. Show that there exists $N \in \mathbb{N}^+$ such that for all $n > N$ then $x_n > 0$.

Answer:

Claim: For $x_n \to x$ where $x > 0$ there exists $N \in \mathbb{N}^+$ such that for $n > N$ then $x_n > 0$.

Proof. For any $\epsilon > 0$ we may find $N \in \mathbb{N}^+$ such that $|x_n - x| < \epsilon$ for all $n > N$, and hence,

$$-\epsilon < x_n - x < \epsilon$$

Choose $\epsilon = \frac{x}{2}$.

(You could choose any value for ϵ in $(0, x)$ you like - I have just picked a half x).

Then for $n > N$ we have,

$$-\frac{x}{2} < x_n - x < \frac{x}{2}$$

so in particular,

$$0 < \frac{x}{2} < x_n$$

□

Question 5.

Let (x_n), (y_n) and (z_n) be sequences. Suppose the first two share the same limit, $x_n \to z$ and $y_n \to z$ as $n \to \infty$. Suppose eventually the terms of the sequence z_n are equal to, or lie between those of x_n and y_n ie. there exists $N \in \mathbb{N}^+$ such that $x_n \leq z_n \leq y_n$ for all $n > N$.

Prove that $z_n \to z$ as $n \to \infty$.

This result is known as the **squeezing theorem**.

Answer:

Claim: Let $x_n \to z$ and $y_n \to z$ as $n \to \infty$. There exists $N \in \mathbb{N}^+$ such that $x_n \leq z_n \leq y_n$ for all $n > N$. Then $z_n \to z$ as $n \to \infty$.

Proof. Let $\epsilon > 0$.

Take $N_1 \in \mathbb{N}^+$ such that $x_n \leq z_n \leq y_n$ for all $n > N_1$.

Take $N_2 \in \mathbb{N}^+$ such that $|x_n - z| < \epsilon$ and $|y_n - z| < \epsilon$ for all $n > N_2$.

Let $N = \max\{N_1, N_2\}$.

Then for $n > N$ we have,

$$-\epsilon < x_n - z < \epsilon \qquad \text{and} \qquad -\epsilon < y_n - z < \epsilon$$

and so since $x_n \leq z_n \leq y_n$ we see,

$$-\epsilon < x_n - z \leq z_n - z \leq y_n - z < \epsilon$$

and hence,

$$|z_n - z| < \epsilon$$

□

Question 6.

Let the sequence (x_n) tend to zero, so $x_n \to 0$ as $n \to \infty$. Consider a sequence (y_n) and suppose there exists $N \in \mathbb{N}^+$ such that for $n > N$ then $|y_n| \leq |x_n|$. Prove that $y_n \to 0$ as $n \to \infty$.

[Hint: use the squeezing theorem.]

Answer:

Claim: Let $x_n \to 0$ as $n \to \infty$. There exists $N \in \mathbb{N}^+$ such that $|y_n| \leq |x_n|$ for all $n > N$. Then $y_n \to 0$ as $n \to \infty$.

Proof. For $n > N$ we have, $|y_n| \leq |x_n|$, so that,

$$-|x_n| \leq y_n \leq |x_n|$$

Now the sequence $a_n = |x_n| \to 0$ as $n \to \infty$, and $b_n = -|x_n| \to 0$ as $n \to \infty$, and for $n > N$ we have $b_n \leq y_n \leq a_n$ so by the squeezing theorem $y_n \to 0$.

□

The above would be sufficient as a proof, but just in case we worry about whether $|x_n| \to 0$ if $x_n \to 0$;

Claim: Let $x_n \to 0$ as $n \to \infty$. Then $|x_n| \to 0$ as $n \to \infty$.

Proof. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}^+$ such that $|x_n - 0| < \epsilon$ for all $n > N$. Hence for all $n > N$ we also have $||x_n| - 0| < \epsilon$.

□

Question 7.

Use the fact that for $x \in \mathbb{R}$ and $x \geq 0$ then $1 - x \leq \cos x \leq 1$ and $1 - x \leq \frac{\sin x}{x} \leq 1$ to prove the sequences,

$$x_n = \cos\left(\frac{1}{n}\right)$$
$$y_n = n \sin\left(\frac{1}{n}\right)$$

both converge to one, so $x_n \to 1$ and $y_n \to 1$ as $n \to \infty$.

Answer:

Claim: Let $x_n = \cos\left(\frac{1}{n}\right)$ and $y_n = n\sin\left(\frac{1}{n}\right)$. Then $x_n \to 1$ and $y_n \to 1$ as $n \to \infty$.

Proof. Define the sequences $a_n = 1 - \frac{1}{n}$ and $b_n = 1$. As $n \to \infty$ then $a_n \to 1$ (from Qu 2.) and $b_n \to 1$. Since, $1 - x \leq \cos(x) \leq 1$ and $1 - x \leq \frac{\sin(x)}{x} \leq 1$ for any $x \geq 0$ then (taking $x = \frac{1}{n}$),

$$a_n \leq x_n \leq b_n \qquad \text{and} \qquad a_n \leq y_n \leq b_n$$

for all $n \in \mathbb{N}^+$. By the squeezing theorem then $x_n \to 1$ and $y_n \to 1$ as $n \to \infty$.

□

Question 8.

Consider the sequence (x_n) for $n \in \mathbb{N}^+$ with

$$x_{n+1} = \sqrt{1 + x_n} \tag{5}$$

and $x_1 = 0$.

Prove that it is bounded above by $x = \frac{1+\sqrt{5}}{2}$ (so that $x = \sqrt{1+x}$).

Prove that it converges. [Hint: show it is increasing.]

Answer:

Claim: x_n defined as above is bounded above by $x = \frac{1+\sqrt{5}}{2}$.

Proof. Note that since $x_1 = 0$ and $x_2 = 1 > 0$ then all $x_n \geq 0$.

Use induction;

- for $n = 1$ the proposition is true; $x_1 \leq x$.
- Suppose for some n that $0 \leq x_n \leq x$. Then,

  $$x - x_{n+1} = x - \sqrt{1 + x_n} \tag{6}$$

  but $\sqrt{1+x_n} \leq \sqrt{1+x}$ (since $0 \leq x_n \leq x$), so,

  $$x - x_{n+1} \geq x - \sqrt{1+x} = 0 \tag{7}$$

  So $x_{n+1} \leq x$ too.

By induction for all $n \geq 1$ we have, $x_n \leq x$. So x_n is bounded by x.

□

Claim: x_n defined as above converges.

Proof. Consider,

$$x_{n+1}^2 - x_n^2 = 1 + x_n - x_n^2 = \frac{5}{4} - \left(x_n - \frac{1}{2}\right)^2 \tag{8}$$

Now the r.h.s is positive or zero for $\frac{1-\sqrt{5}}{2} \leq x_n \leq x = \frac{1+\sqrt{5}}{2}$. Hence for $0 \leq x_n \leq x$ we have $x_{n+1}^2 - x_n^2 \geq 0$ so $x_{n+1} \geq x_n$. The sequence is increasing.

Since the sequence is bounded from above and is increasing, it converges.

□