Mathematical Analysis 2017-8
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Example sheet 5

Series

Question 1.

Prove this comparison test;

Proposition: Let $\sum_{k=1}^{\infty} b_k$ be a divergent series such that $0 \leq b_k$. The series $\sum_{k=1}^{\infty} a_k$ diverges if $b_k \leq a_k$ for all $k \in \mathbb{N}^+$.

Then use this to prove;

Proposition: Let $\sum_{k=1}^{\infty} b_k$ be a divergent series such that $0 \leq b_k$. The series $\sum_{k=1}^{\infty} a_k$ diverges if there exists $N \in \mathbb{N}^+$ such that for all $k > N$ then $b_k \leq a_k$.

Answer:

Proof. (of first claim)

Denote the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ as the sequences (S_n) and (T_n) respectively.

Since $0 \leq a_k$ and $0 \leq b_k$ then (S_n) and (T_n) are increasing sequences. Since $0 \leq b_k \leq a_k$ then $T_n \leq S_n$ for all $n \in \mathbb{N}^+$.

Since $\sum_{k=1}^{\infty} b_k$ is divergent, then T_n is divergent (ie. does not converge).

Assume for contradiction that $\sum_{k=1}^{\infty} a_k$ converges so that (S_n) is convergent. Then let the limit be $S_n \to S$ as $n \to \infty$. Since S_n is increasing, $S_n \leq S$ for all n. Since $T_n \leq S_n$ then $T_n \leq S$. Then (T_n) is an increasing sequence that is bounded above, which implies it converges, leading to a contradiction.

Hence we conclude that (S_n) cannot converge, so $\sum_{k=1}^{\infty} a_k$ is divergent.

□

Proof. (of second claim)

Define new series; $\sum_{k=1}^{\infty} \bar{a}_k$ with $\bar{a}_k = a_{k+N}$ and $\sum_{k=1}^{\infty} \bar{b}_k$ with $\bar{b}_k = b_{k+N}$.

If $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} \bar{b}_k$ diverges (they differ only by the number, $\sum_{k=1}^{N} b_k$).

Now since $0 \leq b_k \leq a_k$ for all $k > N$, then $0 \leq \bar{b}_k \leq \bar{a}_k$ for all $k \in \mathbb{N}^+$, and we can apply the first proposition to conclude that $\sum_{k=1}^{\infty} \bar{a}_k$ diverges.

Then this implies $\sum_{k=1}^{\infty} a_k$ also diverges (since they differ only by the number $\sum_{k=1}^{N} a_k$).

□

Question 2.

Use the comparison test to **prove** whether the following series converge or diverge;

a) $\sum_{k=1}^{\infty} \frac{1}{k^p}$ for $p < 1$

b) $\sum_{k=1}^{\infty} \frac{1}{k!}$

c) $\sum_{k=1}^{\infty} \frac{5^{\frac{1}{k}}}{k}$

d) $\sum_{k=1}^{\infty} \frac{k}{\sqrt{(k+2)(k+5)(k+7)}}$

e) $\sum_{k=1}^{\infty} \frac{k^2}{(k+1)(k+3)2^k}$

f) $\sum_{k=1}^{\infty} \frac{k^2-k+3}{k(4k^3-1)}$

You may use the results from lectures;

- the geometric series $\sum_{k=0}^{\infty} b^k$ is convergent for $|b| < 1$.
- the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p = 1$.

Answer:

Claim: $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{1}{k^p}$ for $p < 1$ is divergent.

Proof. Now for all $k \in \mathbb{N}^+$ we have,

$$a_k = \frac{1}{k^p} > \frac{1}{k}$$

since $p < 1$.

Now, $\sum_{k=1}^{\infty} b_k$ with $b_k = \frac{1}{k}$ diverges, so since $0 < b_k < a_k$ then by comparison $\sum_{k=1}^{\infty} a_k$ diverges.

□

Claim: $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{1}{k!}$ is convergent.

Proof. Now for $k > 1$ then,

$$a_k = \frac{1}{k!} = \frac{1}{k(k-1)(k-2)\dots 1} \leq \frac{1}{k(k-1)} \leq \frac{1}{(k-1)^2}$$

Now, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=2}^{\infty} b_k$ with $b_k = \frac{1}{(k-1)^2}$ converges. Then since $|a_k| \leq b_k$ by the comparison $\sum_{k=1}^{\infty} a_k$ converges.

□

Claim: $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{5^{\frac{1}{k}}}{k}$ is divergent.

Proof. We have,

$$a_k = \frac{5^{\frac{1}{k}}}{k} \geq \frac{1}{k}$$

Now, $\sum_{k=1}^{\infty} b_k$ with $b_k = \frac{1}{k}$ diverges, so since $0 < b_k \leq a_k$ then by comparison $\sum_{k=1}^{\infty} a_k$ diverges.

□

Claim: $\sum_{k=1}^{\infty} \frac{k^2}{(k+1)(k+3)2^k}$ is convergent.

Comment; since $a_k \sim \frac{1}{2^k}$ for $k \to \infty$ so this should converge.

Proof.

$$a_k = \frac{k^2}{(k+1)(k+3)} \frac{1}{2^k} < \frac{1}{2^k}$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, then by the comparison test, $\sum_{k=1}^{\infty} a_k$ converges (as $0 < a_k < \frac{1}{2^k}$). □

Claim: $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{k}{\sqrt{(k+2)(k+5)(k+7)}}$ is divergent.

Comment; we can see the terms tend to $a_k \sim 1/\sqrt{k}$ so this should diverge. How do we prove it carefully? For example, we can show that for very large k then $a_k > 1/k$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges...

Proof.

$$a_k = \frac{k}{\sqrt{(k+2)(k+5)(k+7)}} = \frac{1}{\sqrt{k}} \left(\frac{1}{\sqrt{(1+\frac{2}{k})(1+\frac{5}{k})(1+\frac{7}{k})}} \right)$$

Since, $u_k = \frac{1}{\sqrt{(1+\frac{2}{k})(1+\frac{5}{k})(1+\frac{7}{k})}}$ limits to $u_n \to 1$ as $n \to \infty$, then there exists $N \in \mathcal{N}^+$ such that for $k > N$ then,

$$a_k \geq \frac{1}{k}$$

Then the second proposition in question 1 tells us that since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

□

Claim: $\sum_{k=1}^{\infty} \frac{k^2-k+3}{k(4k^3-1)}$ is convergent.

Comment; since $a_k \sim \frac{1}{4k^2}$ for $k \to \infty$ so this should converge. We can compare it to, for example, $\sum_{k=1}^{\infty} 1/k^{3/2}$ which converges. (We could choose to compare to any $\sum_{k=1}^{\infty} 1/k^p$ with $1 < p < 2$).

Proof.

$$a_k = \frac{1}{4k^2}\frac{1-\frac{1}{k}+\frac{3}{k^2}}{\left(1-\frac{1}{4k^3}\right)}$$

Since $u_k = \frac{1-\frac{1}{k}+\frac{3}{k^2}}{\left(1-\frac{1}{4k^3}\right)} \to 1$ as $k \to \infty$ then there exists N such that for $k > N$ then $a_k < \frac{1}{k^{3/2}}$.

Then since, $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges then by the comparison test $\sum_{k=1}^{\infty} a_k$ converges (as $0 < a_k < \frac{1}{k^{3/2}}$ for all $k > N$).

□

Question 3.

Prove whether the following alternating series converge or diverge;

a) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(1+k^2)}$

b) $\sum_{k=1}^{\infty} \frac{(-1)^k(2+k)}{(1+k)}$

You may use state without proof standard tests.

Answer:

Claim: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(1+k^2)}$ is convergent.

Proof. We have, $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$ with,

$$a_k = \frac{1}{(1+k^2)}$$

and so (a_k) is a decreasing sequence with $a_k \to 0$ as $k \to \infty$. By the alternating series test this converges.

□

Claim: $\sum_{k=1}^{\infty} \frac{(-1)^k(2+k)}{(1+k)}$ is divergent.

Proof. We have $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = \frac{(-1)^k\,(2+k)}{(1+k)}$$

A standard necessary condition for convergence of $\sum_{k=1}^{\infty} a_k$ is that $|a_k| \to 0$ as $k \to \infty$. Since this is not the case, with $|a_k| \to 2$ then the series $\sum_{k=1}^{\infty} a_k$ must diverge.

□

Question 4.

Use the Ratio test to decide on the convergence of these series;

a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

You may use the limit $(1+\frac{1}{n})^n \to e$ as $n \to \infty$.

Answer:

Claim: $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

Proof. We have $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = \frac{k^2}{2^k}$$

Consider,

$$y_k = \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(k+1)^2}{2^{k+1}}}{\frac{k^2}{2^k}}\right| = \frac{(k+1)^2}{2k^2}$$

Then $y_k \to y = \frac{1}{2}$ as $k \to \infty$. Hence from the Ratio test since $y < 1$ the series $\sum_{k=1}^{\infty} a_k$ converges.

□

Claim: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

Proof. We have $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = \frac{k!}{k^k}$$

Consider,

$$y_k = \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}}\right| = \frac{k^k}{(k+1)^{k+1}}(k+1)$$
$$= \frac{k^k}{(k+1)^k} = \frac{1}{(1+\frac{1}{k})^k}$$

We are given in the question that $y_k \to y = \frac{1}{e}$ as $k \to \infty$. Hence from the Ratio test since $y = \frac{1}{e} < 1$ the series $\sum_{k=1}^{\infty} a_k$ converges.

□

Question 5.

Consider the power series

$$\sum_{n=0}^{\infty} \left(2 - (-1)^n\right) 2^n x^n$$

Show that the Ratio test is inconclusive.

Deduce that the Root test shows the series converges absolutely for $|x| < \frac{1}{2}$ and diverges if $|x| > \frac{1}{2}$. What happens for $x = \pm\frac{1}{2}$?

Answer:

Firstly consider the ratio test. We have $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = \left(2 - (-1)^k\right) 2^k x^k$$

Then,

$$y_k = \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\left(2-(-1)^{k+1}\right)2^{k+1}x^{k+1}}{\left(2-(-1)^k\right)2^k x^k}\right|$$
$$= 2|x| \left|\frac{\left(2+(-1)^k\right)}{\left(2-(-1)^k\right)}\right|$$

However we see y_k does not converge as $k \to \infty$, but asymptotically alternates between $6|x|$ and $\frac{2}{3}|x|$.

Secondly consider the root test. Then,

$$y_k = |a_k|^{\frac{1}{k}} = \left|\left(2 - (-1)^k\right) 2^k x^k\right|^{\frac{1}{k}}$$
$$= 2|x| \left|2 - (-1)^k\right|^{\frac{1}{k}}$$

Now recall that since $3^{1/k} \to 1$ as $k \to \infty$ and $1^{1/k} = 1$ then $\left|2 - (-1)^k\right|^{\frac{1}{k}} \to 1$ as $k \to \infty$. Hence,

$$y_k \to y = 2|x|$$

Thus the root test says that for $|x| < \frac{1}{2}$ so $y < 1$ then the series converges. For $|x| > \frac{1}{2}$ it diverges. For $|x| = \frac{1}{2}$ it is inconclusive.

Consider $|x| = +\frac{1}{2}$. Then,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1} \left(2 - (-1)^k\right)$$

A necessary condition for convergence is $|a_k| \to 0$ as $k \to \infty$. So this does not converge (as $|a_k| \not\to 0$).

Consider $|x| = -\frac{1}{2}$. Then,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1} \left(2 - (-1)^k\right) (-1)^k$$

and by the same argument as above this does not converge.

Question 6.

Use the Ratio test to deduce the radius of convergence R of the following two power series. Show that the first converges when $x = \pm R$ and the second diverges for $x = \pm R$.

a) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^2} x^n$

b) $\sum_{n=0}^{\infty} (-1)^n \left(2^n + n^2\right) x^n$

Answer:

a) We have $\sum_{n=0}^{\infty} a_n$ with,

$$a_n = (-1)^n \frac{1}{(n+1)^2} x^n$$

Then,

$$\begin{aligned} y_n = \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}\frac{1}{(n+2)^2}x^{n+1}}{(-1)^n \frac{1}{(n+1)^2}x^n}\right| \\ &= \left|\frac{(n+2)^2}{(n+1)^2}x\right| \to |x| \end{aligned}$$

as $n \to \infty$. Hence the Ratio test implies the series is convergent for $|x| < 1$, and divergent for $|x| > 1$.

For $x = \pm 1$ then,

$$a_n = (\pm 1)^n \frac{1}{(n+1)^2}$$

so,

$$|a_n| = \frac{1}{(n+1)^2} \le \frac{1}{n^2}$$

Since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, then $\sum_{n=0}^{\infty} a_n$ does by the comparison test.

b) We have $\sum_{n=0}^{\infty} a_n$ with,

$$a_n = (-1)^n \left(2^n + n^2\right) x^n$$

Then,

$$\begin{aligned} y_n = \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}\left(2^{n+1}+(n+1)^2\right)x^{n+1}}{(-1)^n (2^n+n^2)x^n}\right| \\ &= \left|\frac{\left(2^{n+1}+(n+1)^2\right)}{(2^n+n^2)}x\right| \to 2|x| \end{aligned}$$

as $n \to \infty$. Hence the Ratio test implies the series is convergent for $|x| < \frac{1}{2}$, and divergent for $|x| > \frac{1}{2}$.

For $x = \pm\frac{1}{2}$ then,

$$|a_n| = \left(1 + \frac{n^2}{2^n}\right)$$

and so $|a_n| \to 1$ as $n \to \infty$. A necessary condition for convergence is $|a_n| \to 0$. Hence the series is divergent for these values.

Question 7.

Consider the series

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n}$$

where $|a_n| < B$. Show that the partial sums of this series are a Cauchy sequence (hence the series converges).

Answer:

Claim: The partial sums $S_n = \sum_{k=0}^{n} \frac{a_k}{2^k}$ are a Cauchy sequence.

Proof. Consider $n > m$ so,

$$S_n - S_m = \sum_{k=m+1}^{n} \frac{a_k}{2^k} = \frac{1}{2^m} \sum_{k=1}^{n-m} \frac{a_{m+k}}{2^k}$$

Hence,

$$\begin{aligned} |S_n - S_m| &= \frac{1}{2^m} \left| \sum_{k=1}^{n-m} \frac{a_{m+k}}{2^k} \right| \\ &\leq \frac{1}{2^m} \sum_{k=1}^{n-m} \left| \frac{a_{m+k}}{2^k} \right| \quad \text{(triangle inequality)} \\ &\leq \frac{B}{2^m} \sum_{k=1}^{n-m} \frac{1}{2^k} \leq \frac{B}{2^m} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{B}{2^m} \frac{1}{1-\frac{1}{2}} = \frac{B}{2^{m-1}} \end{aligned}$$

Let $\epsilon > 0$. We may always choose $N \in \mathbb{N}^+$ such that $B/2^{N-1} < \epsilon$. Then for all $n, m > N$,

$$|S_n - S_m| \leq \max\left\{ \frac{B}{2^{n-1}}, \frac{B}{2^{m-1}} \right\} < \epsilon$$

So the partial sums are a Cauchy sequence.

$\square$