Mathematical Analysis 2017-8

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Example sheet 6

Functions

Question 1.

Sketch the function $f : A \to \mathbb{R}$;

$$f(x) = \begin{cases} 2 & x = 0 \\ e^{-x} & x > 0 \\ x^2 & x < 0 \end{cases}$$

defined on the domain $A = (-1, 1)$. State (without proof) the following;

- $\lim_{x \to \frac{1}{2}} f(x)$
- $\lim_{x \to 0} f(x)$ if it exists - otherwise $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$
- $\sup(f)$ and $\inf(f)$
- The maximum and minimum of f if they exist.

Question 2.

Prove using $\epsilon - \delta$ that the function,

$$\begin{aligned} f : (1, \infty) &\to \mathbb{R} \\ x &\to f(x) = \sqrt{x - 1} \end{aligned}$$

has the limit $\lim_{x \to 1^+} f(x) = 0$.

Question 3.

Suppose we have a function $f : A \to \mathbb{R}$ and a point $y \in A$ such that $l_1 = \lim_{x \to y^-} f(x)$ and $l_2 = \lim_{x \to y^+} f(x)$.

- Prove that if $l = l_1 = l_2$ then $l = \lim_{x \to y} f(x)$.
- Prove that if $l_1 \neq l_2$ then $\lim_{x \to y} f(x)$ does not exist.

Question 4.

Firstly sketch the function $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} 2 - x & x > 0 \\ 1 + x & x \leq 0 \end{cases}$$

Prove using $\epsilon - \delta$ that it has no limit at $x = 0$, ie. $\lim_{x \to 0} f(x)$ does not exist. You may use the result from the previous question.

Question 5.

Sketch the function $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Prove this has no limit at $x = 0$ from the right (ie. $\lim_{x \to 0^+} f(x)$ does not exist).

Question 6.

Assume the functions $f(x)$ and $g(x)$ are defined on $A = (a, b)$ and are continuous at a point $y \in A$. Prove that the product of the functions, $f(x) \cdot g(x)$, is also continuous at y.

[Be careful to treat the situation that $f(y) = 0$ or $g(y) = 0$ properly. This is very similar to when we considered the limit of a product of two sequences.]

Question 7.

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is continuous on all of $\mathbb{R}$.

Hint: Consider a point $y \in \mathbb{R}$ and treat $y = 0$ and $y \neq 0$ separately; in the latter case use, $x^2 - y^2 = (x - y)^2 + 2y\,(x - y)$.

Question 8.

Sketch the function $f : \mathbb{R} \to \mathbb{R}$ with,

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Prove using $\epsilon - \delta$ that it is continuous at $x = 0$.

In fact this function is not only continuous at $x = 0$, but also infinitely differentiable (C^∞) there.

Compute $f^{(n)}(0)$ and hence show $f(x)$ is not analytic at $x = 0$.

[Hint: You are not required to show f is C^∞ at $x = 0$. So to compute $f^{(n)}(0)$ you need only consider $x \leq 0$ - ie. you need not show the derivative computed from $x > 0$ has a limit as $x \to 0^+$ that agrees with that for $x \to 0^-$.]

Question 9.

Compute the Taylor series for $f(x) = \frac{1}{1-x}$ about $x = 0$, and confirm that it gives,

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

Consider this series and hence show that $f(x) = \frac{1}{1-x}$ is analytic at $x = 0$.

Question 10.

Use Taylor's theorem to prove that $f(x) = \cos x$ is an analytic function at $x = 0$.