

M.Phys 1

Mathematical AnalysisSolutions to Problems 3.

$$\begin{aligned}
 \textcircled{1} \quad a) \int x^3 \sin x \, dx &= -\int x^3 d(\cos x) = -x^3 \cos x + 3 \int x^2 \cos x \, dx \\
 &= -x^3 \cos x + 3 \int x^2 d(\sin x) \\
 &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x \, dx \\
 &= -x^3 \cos x + 3x^2 \sin x + 6 \int x d(\cos x) \\
 &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 b) \int \frac{\tan^{-1} x}{x} \, dx &= x \tan^{-1} x - \int x \frac{d}{dx}(\tan^{-1} x) \, dx \\
 &= x \tan^{-1} x - \int \frac{x \, dx}{1+x^2} = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c.
 \end{aligned}$$

$$\textcircled{2} \quad a) \text{ Use } x = \tan \theta \quad \int_0^1 \frac{dx}{(1+x^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2 \theta \, d\theta}{\sec^3 \theta} = \int_0^{\pi/4} \cos \theta \, d\theta = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
 b) \int_0^\infty \frac{dx}{1+e^{2x}} &= \frac{1}{2} \int_1^\infty \frac{du}{u(1+u)} = \frac{1}{2} \int_1^\infty \left(\frac{1}{u} - \frac{1}{1+u} \right) du = \frac{1}{2} \left[\ln \left| \frac{u}{1+u} \right| \right]_1^\infty \\
 &= \frac{1}{2} \left[\ln 1 - \ln \frac{1}{2} \right] = \frac{1}{2} \ln 2.
 \end{aligned}$$

$$\begin{aligned}
 c) \int_1^{3/2} \frac{dx}{(2-x)(x-1)^{3/2}} &= \int_0^{1/\sqrt{2}} \frac{2x \, du}{(1-u^2)^{3/2}} = \int_0^{1/\sqrt{2}} \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du \\
 &= \left[\ln \left| \frac{1+u}{1-u} \right| \right]_0^{1/\sqrt{2}} = \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) = 2 \ln(\sqrt{2}+1).
 \end{aligned}$$

$$\textcircled{3} \quad a) \int_\epsilon^1 \ln x \, dx = [x(\ln x - 1)]_\epsilon^1 = -1 - \epsilon(\ln \epsilon - 1) = \epsilon - 1 - \epsilon \ln \epsilon$$

What does $\epsilon \ln \epsilon$ do as $\epsilon \rightarrow 0$? Actually $\lim_{\epsilon \rightarrow 0} (\epsilon \ln \epsilon) \rightarrow 0$, so
 RHS $\rightarrow -1$ as $\epsilon \rightarrow 0$. $\therefore$ Convergent.

$$b) \int_0^{1-\epsilon} \frac{dx}{(x-1)^2} = -[(x-1)^{-1}]_0^{1-\epsilon} = \frac{1}{\epsilon} - 1. \text{ Not convergent as } \epsilon \rightarrow 0.$$

$$\begin{aligned}
 \textcircled{4} \quad \int x^k \ln x \, dx &= \frac{1}{k+1} \int \ln x \, d(x^{k+1}) = \frac{1}{k+1} \left[(\ln x) x^{k+1} - \int x^{k+1} x^{-1} \, dx \right] \\
 &= \frac{1}{k+1} \left[x^{k+1} \ln x - \frac{1}{k+1} x^{k+1} \right] + c \quad k \neq -1. \\
 &= (k+1)^{-2} [(k+1) \ln x - 1] x^{k+1} + c.
 \end{aligned}$$