PHYSICS 1: MATHEMATICAL ANALYSIS I.

PROBLEMS 7

1. If f and g are any twice-differentiable functions, use the chain rule, along with the new variables $s = x + y$ and $t = x + \frac{1}{2}y$, to show that

$$V(x, y) = f(x + y) + g(x + \tfrac{1}{2}y)$$

satisfies the partial differential equation

$$V_{xx} - 3V_{xy} + 2V_{yy} = 0,$$

where the suffices denote partial derivatives.

2. If $u = u(x, y)$ and x and y transform into two new variables s and t such that $s = \frac{x}{x^2+y^2}$ and $t = \frac{y}{x^2+y^2}$, show that

$$u_s^2 + u_t^2 = \left(u_x^2 + u_y^2\right)\left(x^2 + y^2\right)^2.$$

3. Are the following exact differentials? If so, of what functions?

(i) $e^y dx + x(e^y + 1)dy$; (ii) $(e^y + ye^x)dx + (e^x + xe^y + 1)dy$

More challenging problems

4* If $u = u(x, y)$ and x and y are related to two new independent variables s and t by

$$x = st, \qquad y = \frac{s+t}{s-t},$$

use the chain rule to find $\frac{\partial u}{\partial s}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Solve this to show that

$$2x\frac{\partial u}{\partial x} = s\frac{\partial u}{\partial s} + t\frac{\partial u}{\partial t},$$

and

$$4y\frac{\partial u}{\partial y} = \left(s^2 - t^2\right)\left(\frac{1}{s}\frac{\partial u}{\partial t} - \frac{1}{t}\frac{\partial u}{\partial s}\right).$$