

M. Phys 1 Mathematical AnalysisSolutions to Sheet 8.

a) $u_n = \frac{1}{(n+1)!}$ $\therefore \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+2)!} = \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0$. Convergent

b) $u_n = \frac{(3-4i)^n}{n!}$ $\therefore \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \left| \frac{(3-4i)^{n+1}}{(3-4i)^n} \right| = \lim_{n \rightarrow \infty} \frac{5}{n+1} = 0$. Convergent.

$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x$. Since $\tan x$ is odd, we need to go to x^5 .

Differentiate the diff. equation 4 times: $y'' = 2yy'$; $y''' = 2(y'y'' + y'^2)$

$y^{(4)} = 2(y'y''' + 3y'y''^2)$, $y^{(5)} = 2(y'y^{(4)} + 4y'y''y''' + 3y'^2y'')$.

At $x=0$: $y=0$, $y'=1$, $y''=0$, $y'''=2$, $y^{(4)}=0$, $y^{(5)}=16$.

$\therefore y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \frac{x^5}{5!} y^{(5)}(0) \dots$
 $= 0 + x + 0 + \frac{2}{3!} x^3 + 0 + \frac{16}{5!} x^5 \dots$

$\therefore y = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots$

3) $u_x = \frac{x^2 + 2xy + 2y - y^2 - 1}{(x+y)^2}$; $u_y = \frac{2xy + y^2 - x^2 - 2x - 1}{(x+y)^2}$

$u_x = u_y = 0$ Add to get $2xy = x - y + 1$. Subtract to get $x - y + 1 = 0$ ($x+y \neq 0$).

Together we have $x=0$, $y=1$ and $y=0$, $x=-1$. Two points $(0,1)$, $(-1,0)$.

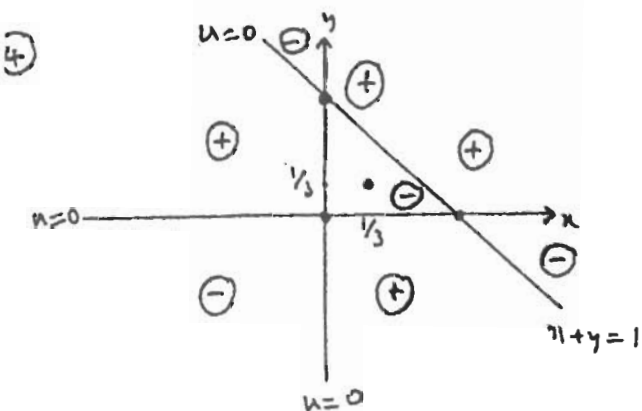
After a lot of work: $u_{xx} = \frac{4y^2 - 4y + 2}{(x+y)^3}$ $u_{yy} = \frac{4x^2 + 4x + 2}{(x+y)^3}$

$u_{xy} = \frac{2(x-y-2xy+1)}{(x+y)^3}$

$(-1,0)$: $u_{xx} = -2$, $u_{yy} = -2$, $u_{xy} = 0$ MAX.

$(0,1)$ $u_{xx} = 2$, $u_{yy} = 2$, $u_{xy} = 0$ MIN.

(Note: At the max $u=0$ while at the min $u=2$. How can this be? Consider that u becomes infinite along the line $y=-x$.)



Signs in circles refer to the sign of u .

$u_x = y(2x+y-1)$, $u_y = x(x+2y-1)$

$\therefore$ Stat. pts at $(0,0)$, $(0,1)$, $(1,0)$ & $(\frac{1}{3}, \frac{1}{3})$.

Consider changes of sign in u across each point. Clearly $(0,0)$, $(0,1)$, $(1,0)$ are SADDLES.

Check $u_{xy}^2 - u_{xx}u_{yy} > 0$. This is 1 for these points, and is $-\frac{1}{3}$ for $(\frac{1}{3}, \frac{1}{3})$. This is a MINIMUM as $u_{xx} > 0$ here.