**Complex Analysis (Prof. M. McCall)**
**Multiple Choice Sheet 2 - Solutions**

1. With time in units of years, the ODE becomes
$$\frac{dP}{dt} = \frac{r}{100}P - Q \ .$$

2. (a) For the three different contributions, look at how they change the volume of the lake.
$$\frac{dV}{dt} = n_i - n_o h - EA \ ,$$
where we have $A = \pi r^2 = \pi\alpha^2 h^2$. As the volume of a cone is given by
$$V = \pi r^2 \frac{h}{3} = \pi\alpha^2 \frac{h^3}{3} \ ,$$
we have
$$\frac{dV}{dh} = \pi\alpha^2 h^2 \ ,$$
and
$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt} = \frac{dV}{dt}\Big/\frac{dV}{dh} = \frac{n_i}{\pi\alpha^2 h^2} - \frac{n_o}{\pi\alpha^2 h} - E \ .$$
(b) The *steady-state* solution is when the differential is zero. Thus we have
$$\frac{n_i}{\pi\alpha^2 h^2} - \frac{n_o}{\pi\alpha^2 h} - E = 0 \ ,$$
which has the solutions
$$h = \frac{-n_0 \pm \sqrt{n_o^2 + 4E\pi\alpha^2 n_i}}{2E\pi\alpha^2} \ .$$
Of the two solutions here we can reject the one corresponding to a negative depth of the lake. After simplification this leads to
$$h = \frac{n_o}{2E\pi\alpha^2}\left(\sqrt{1 + \frac{4E\pi\alpha^2 n_i}{n_o^2}} - 1\right)$$
at very large times. [What happens if $n_0 \to 0$? Can you show that for this case h reverts to the expression given in part (b)ii.?]

3. For each decay it can be written down what adds to it and what takes away from it. We thus have the coupled equations (corresponding to answer (c)):
$$\frac{dN^{\mathrm{Bi}(210)}}{dt} = -(k_\beta^{\mathrm{Bi}(210)} + k_\alpha^{\mathrm{Bi}(210)})N^{\mathrm{Bi}(210)}$$
$$\frac{dN^{\mathrm{Po}(210)}}{dt} = k_\beta^{\mathrm{Bi}(210)} N^{\mathrm{Bi}(210)} - k_\alpha^{\mathrm{Po}(210)} N^{\mathrm{Po}(210)}$$
$$\frac{dN^{\mathrm{Tl}(206)}}{dt} = k_\alpha^{\mathrm{Bi}(210)} N^{\mathrm{Bi}(210)} - k_\beta^{\mathrm{Tl}(206)} N^{\mathrm{Tl}(206)}$$
$$\frac{dN^{\mathrm{Pb}(206)}}{dt} = k_\alpha^{\mathrm{Po}(210)} N^{\mathrm{Po}(210)} + k_\beta^{\mathrm{Tl}(206)} N^{\mathrm{Tl}(206)}$$