Linear Algebra (Vectors)

Combined Problem Set 1 Lectures 1–3

Questions 1–4 are for vectors $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathbb{R}^3$, with $\boldsymbol{a} = 4\,\boldsymbol{i} - 3\,\boldsymbol{k}$, $\boldsymbol{b} = -i + 2\,\boldsymbol{j} - 4\,\boldsymbol{k}$, and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

1. Determine the magnitude of $\boldsymbol{a}$:

   (a) 1 (b) 5 (c) 2.646 (d) 5.099

2. Determine the magnitude of $\boldsymbol{a}+\boldsymbol{c}$:

   (a) $\sqrt{14}$ (b) $5+\sqrt{3}$ (c) $\sqrt{42}$ (d) $\sqrt{7}$ (e) 2

3. Determine if the unit vector $\hat{\boldsymbol{v}} = -\frac{2}{3}\,\boldsymbol{i} + \frac{2}{3}\,\boldsymbol{j} + \frac{1}{3}\,\boldsymbol{k}$ is the unit vector corresponding to any of the following expressions:

   (a) $\frac{1}{4}\,\boldsymbol{a} - \boldsymbol{b}$ (b) $\boldsymbol{c} - \frac{1}{2}\,\boldsymbol{b}$ (c) $2\,\boldsymbol{c} - \boldsymbol{a}$ (d) $\frac{1}{3}\,\boldsymbol{a} - \frac{2}{3}\,\boldsymbol{c}$ (e) None

4. Calculate $\boldsymbol{b}\cdot\boldsymbol{c}$.

5. Consider two vectors in $\mathbb{R}^2$: $\boldsymbol{u} = \boldsymbol{i} + 2\,\boldsymbol{j}$ and $\boldsymbol{v} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$. Determine the values of x and y such that these two vectors are perpendicular to one another.

6. Consider two vectors in $\mathbb{R}^4$: $\boldsymbol{u} = (1,-2,0,2)$ and $\boldsymbol{v} = (2,1,2,x)$. Determine the value of x that makes these two vectors perpendicular.

7. Consider two vectors in $\mathbb{R}^5$: $\boldsymbol{u} = (1,-2,0,3,-1)$ and $\boldsymbol{v} = (-2,0,-4,1,2)$. Determine the angle (in radians) between $\boldsymbol{u}$ and $\boldsymbol{v}$.

8. Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be any three vectors in $\mathbb{R}^3$. Where possible identify equalities between the expressions

   $\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{w})$ $(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w}$ $(\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v}$ $\boldsymbol{u}(\boldsymbol{w}\cdot\boldsymbol{v})$

   and the following expressions:

   $|\boldsymbol{u}||\boldsymbol{w}|\boldsymbol{v}$ $(\boldsymbol{v}\cdot\boldsymbol{u})\boldsymbol{w}$ $(\boldsymbol{w}\cdot\boldsymbol{v})\boldsymbol{u}$ $(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|^2}$ No match

Questions to attempt in your own time Lectures 1–3

1. Consider the following vectors in three-dimensional space $\mathbb{R}^3$:

   $\boldsymbol{a} = 17\,\boldsymbol{i} - 4\,\boldsymbol{j} - \boldsymbol{k} \qquad \boldsymbol{b} = \boldsymbol{j} \qquad \boldsymbol{c} = -2\boldsymbol{i} + 2\,\boldsymbol{j} - \boldsymbol{k} \qquad \boldsymbol{d} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

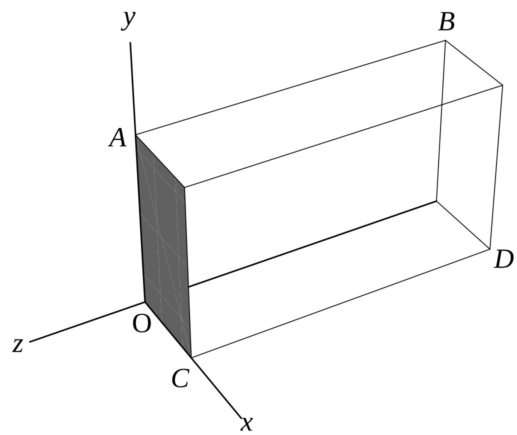
   (a) Write vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in component form.

   (b) Determine the magnitudes of these vectors.

   (c) Find the unit vector corresponding to $\frac{1}{2}\boldsymbol{c} - \boldsymbol{b}$.

   (d) Calculate $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{b} \cdot \boldsymbol{c}$, and $\boldsymbol{c} \cdot \boldsymbol{a}$.

2. In the diagram below,



   the origin is at O, and O, A, B, C, and D are five corners of the rectangular box shown. The position vectors of the points B and C are $\boldsymbol{b} = (0, 8, -14)$ and $\boldsymbol{c} = (4, 0, 0)$, respectively.

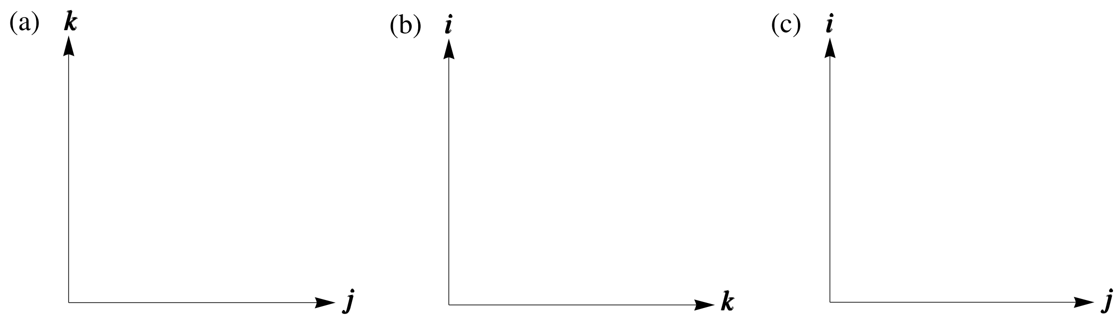
   (a) Determine the vector $\overrightarrow{DA}$ in terms of the basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

   (b) What is the magnitude of $\overrightarrow{DA}$?

3. Let London (L) be at the origin of a two-dimensional right-handed Cartesian coordinate system, where the x-axis points east and the y-axis points north. The coordinates of Mysterytown (M) are $(-88, -66)$ and those of Oxford (O) are $(-80, 28)$. Given that the displacement from Bedford (B) to Oxford is $\overrightarrow{BO} = (-52, -42)$ and the displacement from Bedford to Cambridge (C) is $\overrightarrow{BC} = (42, 11)$, find the displacement $\overrightarrow{CM}$ from Cambridge to Mysterytown. Units are expressed in km. What is the distance – as the crow flies – from Cambridge to Mysterytown? Can you guess the identity of Mysterytown?

Questions for tutorial Lectures 1–3

1. The unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ form a right-handed coordinate set. In each of the following panels, which show two of these three unit vectors, determine whether the third vector points into or out of the plane of the paper.

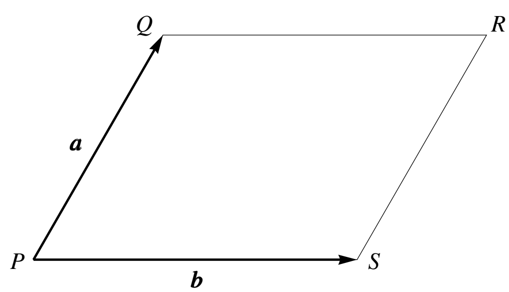


2. For three non-zero real numbers a, b, and c, the equality $ab = ac$ implies that $b = c$. Now suppose that there are three non-zero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\mathbb{R}^n$ that obey

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c}\,.$$

Can you conclude that $\boldsymbol{b} = \boldsymbol{c}$? Justify your answer.

3. In the parallelogram shown below, $\overrightarrow{PQ} = \boldsymbol{a}$ and $\overrightarrow{PS} = \boldsymbol{b}$. Let $\boldsymbol{d}_1 = \overrightarrow{PR}$ and $\boldsymbol{d}_2 = \overrightarrow{SQ}$ be the two diagonals of the parallelogram.



   (a) Express the diagonals $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

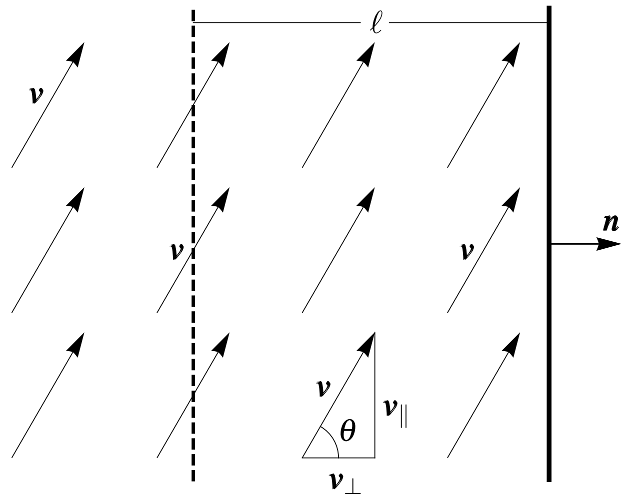
   (b) Use the result of (a) to show that the sum of the squares of the diagonals of a parallelogram is given by the sum of the squares of the four sides.

   (c) By defining M as the midpoint of $\boldsymbol{d}_1$ and point N as the midpoint of $\boldsymbol{d}_2$, find $\overrightarrow{PM}$ and $\overrightarrow{PN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

   (d) What property of a parallelogram can you deduce from (c)?

   (e) Show that the two lines from one corner to the midpoint of the opposite sides trisect the diagonal they cross.

4. Consider a medium in which all particles move with a velocity $\boldsymbol{v}$, as depicted in the figure below:



Assume that the particle density is N particles per unit volume. What is the particle flux through the cross-section A (indicated by the solid vertical line)?

(a) Convince yourself that, for a unit vector $\hat{\boldsymbol{a}}$, the dot product $\boldsymbol{x} \cdot \hat{\boldsymbol{a}}$ is the length of the component of $\boldsymbol{x}$ along the direction specified by $\hat{\boldsymbol{a}}$. Use this to express $|\boldsymbol{v}_{\perp}|$. the length of the velocity component of $\boldsymbol{v}$ that is perpendicular to the plane A in terms of $\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the unit vector normal to the plane.

(b) Calculate the number of particles that will cross A in time Δt. (Referring to the diagram, you might want to consider ℓ, the length scale within which particles will reach the plane in time Δt and the associated volume $V = A\ell$ of particles that will cross the plane within Δt.

(c) If the particles had charge q, what would be the the associated current through the plane.