Linear Algebra (Matrices)

Combined Problem Set 4 Lectures 10–12

Assessed Problems

1. Determine if the following statements are true or false.

(a) $\begin{vmatrix} 9 & -2 & -1 \\ 6 & 1 & -3 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 9 & 6 & 3 \\ -2 & 1 & 2 \\ -1 & -3 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} -4 & 2 & 0 \\ 6 & -2 & -4 \\ -2 & 8 & 4 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 & 0 \\ 3 & -1 & -2 \\ -1 & 4 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 2 & -3 & 4 \\ 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 3 \\ -1 & 3 & 2 & 0 \end{vmatrix} = 0$ (d) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & -3 & 4 & -1 \end{vmatrix} = 0$

[4 marks]

2. Determine if the following statement is true or false.

$$\begin{vmatrix} 2 & -2 & 1 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 7 & -3 & 0 \\ 1 & -2 & 4 & 2 \end{vmatrix}$$

[1 mark]

3. Evaluate

$\begin{vmatrix} 4 & 2 & 1 \\ 7 & 0 & 2 \\ -2 & 0 & 3 \end{vmatrix}$. The choices are: (a) -50, (b) -34, (c) 25, (d) 17.

$\begin{vmatrix} 2 & 14 & -37 & 8 & 11 \\ 0 & 1 & 6 & 23 & -32 \\ 0 & 0 & 4 & 12 & -29 \\ 0 & 0 & 0 & 10 & 20 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$. The choices are: (a)120, (b) 80, (c) 240, (d) 480 .

[2 marks]

4. For each of the following pairs of equations in two unknowns, identify the pairs that have a unique solution.

$$6x+3y = 9, \qquad 1.4x-1.2y = 6.4, \qquad 3x-5y = 8,$$
$$4x+2y = 6. \qquad -2.1x+1.8y = 4.7. \qquad 7x+2y = 12$$

[3 marks]

[Total: 10 marks]

Questions to attempt in your own time Lectures 10–12

1. Determine if the following statement is true or false.

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & -2 & -1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 3 & -2 & -1 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 0 \\ 3 & -2 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

2. Evaluate the determinants

(a) $\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$

How are the determinants related? Evaluate each of thew following determinants,

(c) $\begin{vmatrix} 1 & 5 \\ 4 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 8 & 2 \\ 2 & 5 \end{vmatrix}$ (e) $\begin{vmatrix} 4 & 2 \\ 5 & 7 \end{vmatrix}$

Determine how each of these determinants is developed from those in (a) and (b).

3. Determine which of the following determinants are zero. For each determinant that is zero, identify what characteristic of the determinant ensures that this is so.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 12 \\ -5 & 10 & -15 \end{vmatrix}$ (b) $\begin{vmatrix} 7 & 3 & 2 \\ 6 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & 7 & 0 \\ 3 & -5 & 6 \\ 2 & 3 & -5 \end{vmatrix}$

4. Show that the following determinant is zero by using the general properties of determinants (i.e. without direct evaluation):

$$|\mathsf{A}| = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix},$$

in which a, b, and c are non-zero real numbers. The matrix A, for which $a_{ij} = -a_{ji}$, is called an **anti-symmetric** (or **skew-symmetric**) matrix. Can you show that the value of the determinant is zero for *any* anti-symmetric $n \times n$ matrix when n is an odd number?

1. For which value of y is the value of the following determinant equal to zero?

$$\begin{vmatrix} y-3 & y-9 & y-4 \\ y-6 & y-2 & y \\ y-1 & y+1 & y+8 \end{vmatrix}$$

2. Consider the 2×2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

where a, b, c, and d are real numbers.

   (a) Interchange the rows of this determinant and calculate the result. Interchange the columns, calculate the determinant. Now interchange the rows *and* the columns and calculate the resulting determinant. Comment on you results.

   (b) Add n times the first column to the second column to obtain

$$\begin{vmatrix} a & na+b \\ c & nc+d \end{vmatrix},$$

   Calculate this determinant and comment on your result.

3. Construct a 3×3 matrix with a non-zero determinant and call it A. Then, for each entry of A, compute the corresponding cofactor, and create a new 3×3 matrix with these cofactors by placing the cofactor of an entry in the same location as the entry on which it was based. Call this matrix B. Compute the matrix product $\mathsf{A}\mathsf{B}^{\mathsf{T}}$, where ‘T’ indicates the transpose of a matrix, that is, where $b_{ij} \to b_{ji}$. Comment on your result.

4. Consider the determinant of a (lower) triangular matrix:

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Compute thus determinant and comments on your results.