Proof question pack

June 2018 9MA0 Paper 2

3. (a) "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."

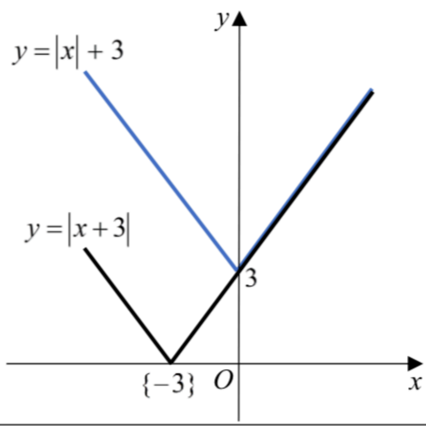
Disprove this statement by means of a counter example.

(2)

(b) (i) Sketch the graph of $y = |x| + 3$

(ii) Explain why $|x| + 3 \geqslant |x + 3|$ for all real values of x.

(3)

Question	Scheme		Marks	AOs
3	Statement: "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."			
(a)	E.g. $m = \sqrt{3}, n = \sqrt{12}$		M1	1.1b
	$\{mn =\} \quad (\sqrt{3})(\sqrt{12}) = 6$ $\Rightarrow$ statement untrue **or** 6 is not irrational or 6 is rational		A1	2.4
			(2)	
(b)(i), (ii) Way 1	 	V shaped graph {reasonably} symmetrical about the y-axis with vertical interpret $(0, 3)$ or 3 stated or marked on the positive y-axis	B1	1.1b
		Superimposes the graph of $y = \lvert x + 3 \rvert$ on top of the graph of $y = \lvert x \rvert + 3$	M1	3.1a
	the graph of $y = \lvert x \rvert + 3$ is either the same or above the graph of $y = \lvert x + 3 \rvert$ {for corresponding values of x} **or** when $x \geq 0$, both graphs are equal (or the same) when $x < 0$, the graph of $y = \lvert x \rvert + 3$ is above the graph of $y = \lvert x + 3 \rvert$		A1	2.4
			(3)	
(b)(ii) Way 2	**<u>Reason 1</u>** When $x \geq 0$, $\lvert x \rvert + 3 = \lvert x + 3 \rvert$ **<u>Reason 2</u>** When $x < 0$, $\lvert x \rvert + 3 > \lvert x + 3 \rvert$	Any one of Reason 1 or Reason 2	M1	3.1a
		Both Reason 1 and Reason 2	A1	2.4
				(5 marks)

Notes for Question 3	
(a)	
M1:	States or uses any pair of ***different*** numbers that will disprove the statement. E.g. $\sqrt{3},\ \sqrt{12}$; $\sqrt{2},\ \sqrt{8}$; $\sqrt{5},-\sqrt{5}$; $\frac{1}{\pi}, 2\pi$; $3e, \frac{4}{5e}$;
A1:	Uses correct reasoning to disprove the given statement, with a correct conclusion
Note:	Writing $(3e)\left(\frac{4}{5e}\right)=\frac{12}{5} \Rightarrow$ untrue is sufficient for M1A1
(b)(i)	
B1:	See scheme
(b)(ii)	
M1:	For constructing a method of comparing $\lvert x\rvert+3$ with $\lvert x+3\rvert$. See scheme.
A1:	Explains fully why $\lvert x\rvert+3 \ge \lvert x+3\rvert$. See scheme.
Note:	Do not allow either $x>0,\ \lvert x\rvert+3 \ge \lvert x+3\rvert$ or $x \ge 0,\ \lvert x\rvert+3 \ge \lvert x+3\rvert$ as a valid reason
Note	$x=0$ (or where necessary $x=-3$) need to be considered in their solutions for A1
Note:	Do not allow an incorrect statement such as $x \le 0,\ \lvert x\rvert+3 > \lvert x+3\rvert$ for A1

Notes for Question 3 Continued	
(b)(ii)	
Note:	Allow M1A1 for $x>0,\ \lvert x\rvert+3=\lvert x+3\rvert$ and for $x \le 0,\ \lvert x\rvert+3 \ge \lvert x+3\rvert$
Note:	Allow M1 for any of • x is positive, $\lvert x\rvert+3=\lvert x+3\rvert$ • x is negative, $\lvert x\rvert+3>\lvert x+3\rvert$ • $x>0,\ \lvert x\rvert+3=\lvert x+3\rvert$ • $x\le 0,\ \lvert x\rvert+3\ge\lvert x+3\rvert$ • $x>0,\ \lvert x\rvert+3$ and $\lvert x+3\rvert$ are equal • $x\ge 0,\ \lvert x\rvert+3$ and $\lvert x+3\rvert$ are equal • when $x\ge 0$, both graphs are equal • for positive values $\lvert x\rvert+3$ and $\lvert x+3\rvert$ are the same Condone for M1 • $x\le 0,\ \lvert x\rvert+3>\lvert x+3\rvert$ • $x<0,\ \lvert x\rvert+3\ge\lvert x+3\rvert$

(b)(ii) Way 3	• For $x>0,\ \lvert x\rvert+3=\lvert x+3\rvert$ • For $-3<x<0$, as $\lvert x\rvert+3>3$ and $\{0<\}\ \lvert x+3\rvert<3$, then $\lvert x\rvert+3>\lvert x+3\rvert$	M1	3.1a
	• For $x\le -3$, as $\lvert x\rvert+3=-x+3$ and $\lvert x+3\rvert=-x-3$, then $\lvert x\rvert+3>\lvert x+3\rvert$	A1	2.4

2. (i) Show that $x^2 - 8x + 17 > 0$ for all real values of x

(3)

(ii) "If I add 3 to a number and square the sum, the result is greater than the square of the original number."

State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

Question	Scheme	Marks	AOs
2(i)	$x^2-8x+17=(x-4)^2-16+17$	M1	3.1a
	$=(x-4)^2+1$ with comment (see notes)	A1	1.1b
	As $(x-4)^2\geqslant 0 \Rightarrow (x-4)^2+1\geqslant 1$ hence $x^2-8x+17>0$ for all x	A1	2.4
		(3)	
(ii)	For an explanation that it may not always be true Tests say $x=-5$ $(-5+3)^2=4$ whereas $(-5)^2=25$	M1	2.3
	States sometimes true and gives reasons Eg. when $x=5$ $(5+3)^2=64$ whereas $(5)^2=25$ True When $x=-5$ $(-5+3)^2=4$ whereas $(-5)^2=25$ Not true	A1	2.4
		(2)	
			(5 marks)

Notes

(i) Method One: Completing the Square

M1: For an attempt to complete the square. Accept $(x-4)^2$ ...

A1: For $(x-4)^2+1$ with either $(x-4)^2\geqslant 0, (x-4)^2+1\geqslant 1$ or min at (4,1). Accept the inequality statements in words. Condone $(x-4)^2>0$ or a squared number is always positive for this mark.

A1: A fully written out solution, with correct statements and no incorrect statements. There must be a valid reason and a conclusion

....

$x^2-8x+17$

$=(x-4)^2+1\geqslant 1$ as $(x-4)^2\geqslant 0$

Hence $(x-4)^2+1>0$

scores M1 A1 A1

....

$x^2-8x+17>0$

$(x-4)^2+1>0$

This is true because $(x-4)^2\geqslant 0$ and when you add 1 it is going to be positive

scores M1 A1 A1

....

$x^2-8x+17>0$

$(x-4)^2+1>0$

which is true because a squared number is positive

scores M1 A1 A0

incorrect and incomplete

....

$x^2-8x+17=(x-4)^2+1$ — scores M1 A1 A0

Minimum is (4,1) so $x^2-8x+17>0$ — correct but not explained

....

$x^2-8x+17=(x-4)^2+1$ — scores M1 A1 A1

Minimum is (4,1) so as $1>0 \Rightarrow x^2-8x+17>0$ — correct and explained

....

$x^2 - 8x + 17 > 0$

$(x-4)^2 + 1 > 0$

scores M1 A0 (no explanation) A0

....

Method Two: Use of a discriminant

M1: Attempts to find the discriminant $b^2 - 4ac$ with a correct *a*, *b* and *c* which may be within a quadratic formula. You may condone missing brackets.

A1: Correct value of $b^2 - 4ac = -4$ **and** states or shows curve is U shaped (or intercept is (0,17)) or equivalent such as +ve x^2 etc

A1: Explains that as $b^2 - 4ac < 0$, there are no roots, and curve is U shaped then $x^2 - 8x + 17 > 0$

....

Method Three: Differentiation

M1: Attempting to differentiate and finding the turning point. This would involve attempting to find $\frac{dy}{dx}$, then setting it equal to 0 and solving to find the *x* value and the *y* value.

A1: For differentiating $\frac{dy}{dx} = 2x - 8 \Rightarrow (4,1)$ is the **turning point**

A1: Shows that (4,1) is the minimum point (second derivative or U shaped), hence $x^2 - 8x + 17 > 0$

....

Method 4: Sketch graph using calculator

M1: Attempting to sketch $y = x^2 - 8x + 17$, U shape with minimum in quadrant one

A1: As above with minimum at (4,1) marked

A1: Required to state that quadratics only have one turning point and as "1" is above the *x*-axis then $x^2 - 8x + 17 > 0$

(ii)

Numerical approach

Do not allow any marks if the student just mentions "positive" and "negative" numbers. Specific examples should be seen calculated if a numerical approach is chosen.

M1: Attempts a value (where it is not true) and shows/implies that it is not true for that value. For example, for -4 : $(-4+3)^2 > (-4)^2$ and indicates not true (states not true, ✗)

or writing $(-4+3)^2 < (-4)^2$ is sufficient to imply that it is not true

A1: Shows/implies that it can be true for a value **AND** states sometimes true. For example for +4 : $(4+3)^2 > 4^2$ and indicates true ✓

or writing $(4+3)^2 > 4^2$ is sufficient to imply this is true following $(-4+3)^2 < (-4)^2$

condone incorrect statements following the above such as 'it is only true for positive numbers' as long as they state "sometimes true" and show both cases.

Algebraic approach

M1: Sets the problem up algebraically Eg. $(x+3)^2 > x^2 \Rightarrow x > k$ Any inequality is fine. You may condone one error for the method mark. Accept $(x+3)^2 > x^2 \Rightarrow 6x + 9 > 0$ oe

A1: States sometimes true a**nd** states/implies true for $x > -\frac{3}{2}$ or states/implies not true for $x \leq -\frac{3}{2}$ In both cases you should expect to see the statement "sometimes true" to score the A1

8. (i) Show that $y^2 - 4y + 7$ is positive for all real values of y.

(2)

(ii) Bobby claims that

$$e^{3x} \geqslant e^{2x} \qquad x \in \mathbb{R}$$

Determine whether Bobby's claim is always true, sometimes true or never true, justifying your answer.

(2)

(iii) Elsa claims that

'for $n \in \mathbb{Z}^+$, if n^2 is even, then n must be even'

Use proof by contradiction to show that Elsa's claim is true.

(2)

(iv) Ying claims that

'the sum of two different irrational numbers is irrational'

Determine whether Ying's claim is always true, sometimes true or never true, justifying your answer.

(2)

Question	Scheme	Marks	AOs
8 (i)	E.g. $y^2 - 4y + 7 = (y-2)^2 - 4 + 7$	M1	2.1
	$= (y-2)^2 + 3 \geqslant 3,$ as $(y-2)^2 \geqslant 0$ and so $y^2 - 4y + 7$ is positive for all real values of *y*	A1	2.2a
		(2)	
(ii)	For an explanation or statement to show when (Bobby's) claim $e^{3x} \geqslant e^{2x}$ fails. This could be e.g. • when $x = -1,$ $e^{-3} < e^{-2}$ **or** e^{-3} is not greater than or equal to e^{-2} • when $x < 0,$ $e^{3x} < e^{2x}$ **or** e^{3x} is not greater than or equal to e^{2x}	M1	2.3
	Followed by an explanation or statement to show when (Bobby's) claim $e^{3x} \geqslant e^{2x}$ is true. This could be e.g. • $x = 2,$ $e^6 \geqslant e^4$ **or** e^6 is greater than or equal to e^4 • when $x \geqslant 0,$ $e^{3x} \geqslant e^{2x}$ **and** a correct conclusion. E.g. • (Bobby's) claim is sometimes true	A1	2.4
		(2)	
(ii) **Alt 1**	Assuming $e^{3x} \geqslant e^{2x}$, then $\ln(e^{3x}) \geqslant \ln(e^{2x}) \Rightarrow 3x \geqslant 2x \Rightarrow x \geqslant 0$	M1	2.3
	Correct algebra, using logarithms, leading from $e^{3x} \geqslant e^{2x}$ to $x \geqslant 0$ **and** a correct conclusion. E.g. (Bobby's) claim is sometimes true	A1	2.4
(iii)	Assume that n^2 is even and *n* is odd. So $n = 2k+1,$ where *k* is an integer.	M1	2.1
	$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ So n^2 is odd which contradicts n^2 is even. So (Elsa's) claim is true.	A1	2.4
		(2)	
(iv)	For an explanation or statement to show when (Ying's) claim "the sum of two different irrational numbers is irrational" fails This could be e.g. • $\pi, 9-\pi;$ sum $= \pi + 9 - \pi = 9$ is not irrational	M1	2.3
	Followed by an explanation or statement to show when (Ying's) claim "the sum of two different irrational numbers is irrational" is true. This could be e.g. • $\pi, 9+\pi;$ sum $= \pi + 9 + \pi = 2\pi + 9$ is irrational and a correct conclusion. E.g.	A1	2.4

Question 8 Notes:	
(i)	
M1:	Attempts to • complete the square **or** • find the minimum by differentiation **or** • draw a graph of $f(y) = y^2 - 4y + 7$
A1:	Completes the proof by showing $y^2 - 4y + 7$ is positive for all real values of *y* with no errors seen in their working.
(ii)	
M1:	See scheme
A1:	See scheme
(ii)	
Alt 1	
M1:	Assumes e^{3x} Ö e^{2x}, takes logarithms and rearranges to make *x* the subject of their inequality
A1:	See scheme
(iii)	
M1:	Begins the proof by negating Elsa's claim and attempts to define *n* as an odd number
A1:	Shows $n^2 = 4k^2 + 4k + 1$, where *n* is correctly defined and gives a correct conclusion
(iv)	
M1:	See scheme
A1:	See scheme

	• (Ying's) claim is sometimes true		
		(2)	
			(8 marks)

11. (a) Prove that for all positive values of x and y

$$\sqrt{xy} \leqslant \frac{x+y}{2}$$

(2)

(b) Prove by counter example that this is not true when x and y are both negative.

(1)

(Total for Question 11 is 3 marks)

Question	Scheme	Marks	AOs
11 (a) **Way 1**	Since x and y are positive, their square roots are real and so $(\sqrt{x}-\sqrt{y})^2 \geq 0$ giving $x - 2\sqrt{x}\sqrt{y} + y \geq 0$	M1	2.1
	$\therefore\ 2\sqrt{xy} \leq x + y$ provided x and y are positive and so $\sqrt{xy} \leq \dfrac{x+y}{2}$ *	A1*	2.2a
		(2)	
Way 2 **Longer method**	Since $(x-y)^2 \geq 0$ for real values of x and y, $x^2 - 2xy + y^2 \geq 0$ and so $4xy \leq x^2 + 2xy + y^2$ i.e. $4xy \leq (x+y)^2$	M1	2.1
	$\therefore\ 2\sqrt{xy} \leq x + y$ provided x and y are positive and so $\sqrt{xy} \leq \dfrac{x+y}{2}$ *	A1*	2.2a
		(2)	
(b)	Let $x = -3$ and $y = -5$ then LHS = $\sqrt{15}$ and RHS= -4 so as $\sqrt{15} > -4$ result does not apply	B1	2.4
		(1)	
			(3 marks)

Notes:

(a)

M1: Need two stages of the three stage argument involving the three stages, squaring, square rooting terms and rearranging

A1*: Need all three stages making the correct deduction to achieve the printed result

(b)

B1: Chooses two negative values and substitutes, then states conclusion

6. Complete the table below. The first one has been done for you.

For each statement you must state if it is always true, sometimes true or never true, giving a reason in each case.

Statement	Always True	Sometimes True	Never True	Reason
The quadratic equation $ax^2 + bx + c = 0, \quad (a \neq 0)$ has 2 real roots.		✓		It only has 2 real roots when $b^2 - 4ac > 0$. When $b^2 - 4ac = 0$ it has 1 real root and when $b^2 - 4ac < 0$ it has 0 real roots.
(i) When a real value of x is substituted into $x^2 - 6x + 10$ the result is positive. **(2)**				
(ii) If $ax > b$ then $x > \dfrac{b}{a}$ **(2)**				
(iii) The difference between consecutive square numbers is odd. **(2)**				

(Total for Question 6 is 6 marks)

Question	Scheme	Marks	AOs
6(i)	$x^2-6x+10=(x-3)^2+1$	M1	2.1
	Deduces "always true" as $(x-3)^2\geqslant 0 \Rightarrow (x-3)^2+1\geqslant 1$ and so is always positive	A1	2.2a
		(2)	
(ii)	For an explanation that it need not (always) be true This could be if $a<0$ then $ax>b\Rightarrow x<\frac{b}{a}$	M1	2.3
	States 'sometimes' and explains if $a>0$ then $ax>b\Rightarrow x>\frac{b}{a}$ if $a<0$ then $ax>b\Rightarrow x<\frac{b}{a}$	A1	2.4
		(2)	
(iii)	Difference $=(n+1)^2-n^2=2n+1$	M1	3.1a
	Deduces "Always true" as $2n+1$ = (even +1) = odd	A1	2.2a
		(2)	
		(6 marks)	

Notes:

(i)

M1: Attempts to complete the square or any other valid reason. Allow for a graph of $y=x^2-6x+10$ or an attempt to find the minimum by differentiation

A1: States always true with a valid reason for their method

(ii)

M1: For an explanation that it need not be true (sometimes). This could be if $a<0$ then $ax>b\Rightarrow x<\frac{b}{a}$ or simply $-3x>6\Rightarrow x<-2$

A1: Correct statement (sometimes true) and explanation

(iii)

M1: Sets up the proof algebraically.
For example by attempting $(n+1)^2-n^2=2n+1$ or $m^2-n^2=(m-n)(m+n)$ with $m=n+1$

A1: States always true with reason and proof
Accept a proof written in words. For example
If integers are consecutive, one is odd and one is even
When squared $\text{odd}\times\text{odd}=\text{odd}$ and $\text{even}\times\text{even}=\text{even}$
The difference between odd and even is always odd, hence always true
Score M1 for two of these lines and A1 for a good proof with all three lines or equivalent.

(Pearson) Practice Paper A

1. It is suggested that the sequence $a_k = 2^k + 1$, $k \geq 1$, produces only prime numbers.

(a) Show that a_1, a_2 and a_4 produce prime numbers.

(2 marks)

(b) Prove by counter example that the sequence does not always produce a prime number.

(2 marks)

1	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
(a)	Makes an attempt to substitute $k = 1$, $k = 2$ and $k = 4$ into $a_k = 2^k + 1, k$ Ö 1	**M1**	1.1b	5th Understand disproof by counter example.
	Shows that $a_1 = 3$, $a_2 = 5$ and $a_4 = 17$ and these are prime numbers.	**A1**	1.1b	
		(2)		
(b)	Substitutes a value of k that does not yield a prime number. For example, $a_3 = 9$ or $a_5 = 33$	**A1**	1.1b	5th Understand disproof by counter example.
	Concludes that their number is not prime. For example, states that $9 = 3 \times 3$, so 9 is not prime.	**B1**	2.4	
		(2)		
				(4 marks)

10. Use proof by contradiction to show that, given a rational number a and an irrational number b, $a - b$ is irrational.

(4 marks)

10	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: given a rational number a and an irrational number b, assume that $a - b$ is rational.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Sets up the proof by defining the different rational and irrational numbers. The choice of variables does not matter. Let $a = \frac{m}{n}$ As we are assuming $a - b$ is rational, let $a - b = \frac{p}{q}$ So $a - b = \frac{p}{q} \Rightarrow \frac{m}{n} - b = \frac{p}{q}$	**M1**	2.2a	
	Solves $\frac{m}{n} - b = \frac{p}{q}$ to make b the subject and rewrites the resulting expression as a single fraction: $\frac{m}{n} - b = \frac{p}{q} \Rightarrow b = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$	**M1**	1.1b	
	Makes a valid conclusion. $b = \frac{mq - pn}{nq}$, which is rational, contradicts the assumption b is an irrational number. Therefore the difference of a rational number and an irrational number is irrational.	**B1**	2.4	
				(4 marks)

Practice Paper B

1. Use proof by contradiction to prove the statement: ‘The product of two odd numbers is odd.’

(5 marks)

1	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: there exists a product of two odd numbers that is even.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines two odd numbers. Can choose any two different variables. 'Let $2m + 1$ and $2n + 1$ be our two odd numbers.'	**B1**	2.2a	
	Successfully multiplies the two odd numbers together: $(2m+1)(2n+1) \equiv 4mn+2m+2n+1$	**M1**	1.1b	
	Factors the expression and concludes that this number must be odd. $4mn+2m+2n+1 \equiv 2(2mn+m+n)+1$ $2(2mn+m+n)$ is even, so $2(2mn+m+n)+1$ must be odd.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption that the product of two odd numbers is even, therefore the product of two odd numbers is odd.	**B1**	2.4	
				(5 marks)

Notes

Alternative method

Assume the opposite is true: there exists a product of two odd numbers that is even. **(B1)**
If the product is even then 2 is a factor. **(B1)**
So 2 is a factor of at least one of the two numbers. **(M1)**
So at least one of the two numbers is even. **(M1)**
This contradicts the statement that both numbers are odd. **(B1)**

2. (a) Prove that the sum of the first n terms of an arithmetic series is $S = \frac{n}{2}\left(2a + \left(n-1\right)d\right)$.

(3 marks)

(b) Hence, or otherwise, find the sum of the first 200 odd numbers.

(2 marks)

2	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
(a)	Writes out the first n terms of the arithmetic sequence in both ascending and descending form $S = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$ $S = (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \ldots + a$	M1	2.4	5th Understand the proof of the Sn formula for arithmetic series.
	Attempts to add these two sequences $2S = (2a+(n-1)d) \times n$	M1	2.4	
	States $S = \frac{n}{2}(2a+(n-1)d)$	A1	1.1b	
		(3)		
(b)	Makes an attempt to find the sum. For example, $S = \frac{200}{2}(2+199(2))$ is seen.	M1	2.2a	4th Understand simple arithmetic series.
	States correct final answer. $S = 40\,000$	A1	1.1b	
		(2)		
				(5 marks)

Notes

(a) Do not award full marks for an incomplete proof.

(a) Do award second method mark if student indicates that $(2a + (n-1)d$ appears n times.

Practice Paper C

5. Prove by contradiction that if n is odd, $n^3 + 1$ is even.

(5 marks)

5	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: there exists a number n such that n is odd and $n^3 + 1$ is also odd.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines an odd number. 'Let $2k + 1$ be an odd number.'	**B1**	2.2a	
	Successfully calculates $(2k+1)^3+1$ $(2k+1)^3+1 \equiv \left(8k^3+12k^2+6k+1\right)+1 \equiv 8k^3+12k^2+6k+2$	**M1**	1.1b	
	Factors the expression and concludes that this number must be even. $8k^3+12k^2+6k+2 \equiv 2\left(4k^3+6k^2+3k+1\right)$ $2\left(4k^3+6k^2+3k+1\right)$ is even.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption that there exists a number n such that n is odd and $n^3 + 1$ is also odd, so if n is odd, then $n^3 + 1$ is even.	**B1**	2.4	
				(5 marks)

Notes

Alternative method

Assume the opposite is true: there exists a number n such that n is odd and $n^3 + 1$ is also odd. (**B1**)
If $n^3 + 1$ is odd, then n^3 is even. (**B1**)
So 2 is a factor of n^3. (**M1**)
This implies 2 is a factor of n. (**M1**)
This contradicts the statement n is odd. (**B1**)

10. Use proof by contradiction to show that there are no positive integer solutions to the statement $x^2 - y^2 = 1$.

(5 marks)

10	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: there exist positive integer solutions to the statement $x^2 - y^2 = 1$'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Sets up the proof by factorising $x^2 - y^2$ and stating $(x-y)(x+y)=1$	**M1**	2.2a	
	States that there is only one way to multiply to make 1: $1\times1=1$ and concludes this means that: $x-y=1$ $x+y=1$	**M1**	1.1b	
	Solves this pair of simultaneous equations to find the values of x and y: $x = 1$ and $y = 0$	**M1**	1.1b	
	Makes a valid conclusion. $x = 1$, $y = 0$ are not both positive integers, which is a contradiction to the opening statement. Therefore there do not exist positive integers x and y such that $x^2 - y^2 = 1$	**B1**	2.4	
				(5 marks)

Practice Paper D

2. (a) Use proof by contradiction to show that if n^2 is an even integer then n is also an even integer.

(4 marks)

(b) Prove that √2 is irrational.

(6 marks)

2	**Scheme**	**Marks**	**AOs**	**Pearson Progression Step and Progress descriptor**
(a)	Begins the proof by assuming the opposite is true. 'Assumption: there exists a number n such that n^2 is even and n is odd.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Defines an odd number (choice of variable is not important) and successfully calculates n^2 Let $2k + 1$ be an odd number. $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$	**M1**	2.2a	
	Factors the expression and concludes that this number must be odd. $4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$, so n^2 is odd.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption n^2 is even. Therefore if n^2 is even, n must be even.	**B1**	2.4	
		(4)		

Practice Paper E

1. Prove by exhaustion that $1+2+3+...+n \equiv \frac{n(n+1)}{2}$ for positive integers from 1 to 6 inclusive.

(3 marks)

Q1	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
1	Makes an attempt to substitute any of n = 1, 2, 3, 4, 5 or 6 into $\frac{n(n+1)}{2}$	M1	1.1b	5th Complete proofs by exhaustion.
	Successfully substitutes n = 1, 2, 3, 4, 5 **and** 6 into $\frac{n(n+1)}{2}$ $1=\frac{(1)(2)}{2}$ $1+2=\frac{(2)(3)}{2}$ $1+2+3=\frac{(3)(4)}{2}$ $1+2+3+4=\frac{(4)(5)}{2}$ $1+2+3+4+5=\frac{(5)(6)}{2}$ $1+2+3+4+5+6=\frac{(6)(7)}{2}$	A1	1.1b	
	Draws the conclusion that as the statement is true for all numbers from 1 to 6 inclusive, it has been proved by exhaustion.	B1	2.4	
				(3 marks)

2. (a) When θ is small, show that the equation $\dfrac{1+\sin\theta+\tan 2\theta}{2\cos 3\theta-1}$ can be written as $\dfrac{1}{1-3\theta}$.

(4 marks)

(b) Hence write down the value of $\dfrac{1+\sin\theta+\tan 2\theta}{2\cos 3\theta-1}$ when θ is small.

(1 mark)

2	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
(a)	Shows that $2\cos 3\theta \approx 2\left(1-\frac{9\theta^2}{2}\right)=2-9\theta^2$	M1	2.1	6th Understand small-angle approximations for sin, cos and tan (angle in radians).
	Shows that $2\cos 3\theta - 1 \approx 1-9\theta^2=\left(1-3\theta\right)\left(1+3\theta\right)$	M1	1.1b	
	Shows $1+\sin\theta+\tan 2\theta = 1+\theta+2\theta=1+3\theta$	M1	2.1	
	Recognises that $\frac{1+\sin\theta+\tan 2\theta}{2\cos 3\theta - 1}\approx\frac{1+3\theta}{\left(1-3\theta\right)\left(1+3\theta\right)}=\frac{1}{1-3\theta}$	A1	1.1b	
		(4)		
(b)	When θ is small, $\frac{1}{1-3\theta}\approx 1$	A1	1.1b	7th Use small-angle approximations to solve problems.
		(1)		
				(5 marks)

10. Use proof by contradiction to show that there is no greatest positive rational number.

(4 marks)

Q	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
10	Begins the proof by assuming the opposite is true. ‘Assumption: there exists a rational number $\frac{a}{b}$ such that $\frac{a}{b}$ is the greatest positive rational number.’	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Makes an attempt to consider a number that is clearly greater than $\frac{a}{b}$: ‘Consider the number $\frac{a}{b}+1$, which must be greater than $\frac{a}{b}$ ’	**M1**	2.2a	
	Simplifies $\frac{a}{b}+1$ and concludes that this is a rational number. $\frac{a}{b}+1 \equiv \frac{a}{b}+\frac{b}{b} \equiv \frac{a+b}{b}$ By definition, $\frac{a+b}{b}$ is a rational number.	**M1**	1.1b	
	Makes a valid conclusion. This contradicts the assumption that there exists a greatest positive rational number, so we can conclude that there is not a greatest positive rational number.	**B1**	2.4	
				(4 marks)

Practice Paper F

2. Use proof by contradiction to show that there exist no integers a and b for which $25a + 15b = 1$.

(4 marks)

2	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. ‘Assumption: there do exist integers *a* and *b* such that $25a+15b=1$’	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Understands that $25a+15b=1 \Rightarrow 5a+3b=\frac{1}{5}$ ‘As both 25 and 15 are multiples of 5, divide both sides by 5 to leave $5a+3b=\frac{1}{5}$’	**M1**	2.2a	
	Understands that if *a* and *b* are integers, then 5*a* is an integer, 3*b* is an integer and 5*a* + 3*b* is also an integer.	**M1**	1.1b	
	Recognises that this contradicts the statement that $5a+3b=\frac{1}{5}$, as $\frac{1}{5}$ is not an integer. Therefore there do not exist integers *a* and *b* such that $25a+15b=1$’	**B1**	2.4	
				(4 marks)

7. Prove by contradiction that there are infinitely many prime numbers.

7	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
	Begins the proof by assuming the opposite is true. 'Assumption: there is a finite amount of prime numbers.'	**B1**	3.1	7th Complete proofs using proof by contradiction.
	Considers what having a finite amount of prime numbers means by making an attempt to list them: Let all the prime numbers exist be $p_1, p_2, p_3, ...p_n$	**M1**	2.2a	
	Consider a new number that is one greater than the product of all the existing prime numbers: Let $N = (p_1 \times p_2 \times p_3 \times ... \times p_n) + 1$	**M1**	1.1b	
	Understands the implication of this new number is that division by any of the existing prime numbers will leave a remainder of 1. So none of the existing prime numbers is a factor of *N*.	**M1**	1.1b	
	Concludes that either *N* is prime or *N* has a prime factor that is not currently listed.	**B1**	2.4	
	Recognises that either way this leads to a contradiction, and therefore there is an infinite number of prime numbers.	**B1**	2.4	
				(6 marks)

Notes

If *N* is prime, it is a new prime number separate to the finite list of prime numbers, $p_1, p_2, p_3, ...p_n$.

If *N* is divisible by a previously unknown prime number, that prime number is also separate to the finite list of prime numbers.

11. (a) Prove that $\dfrac{\tan x - \sec x}{1 - \sin x} \equiv -\sec x,\ \ x \neq (2n+1)\dfrac{\pi}{2}$.

(3 marks)

(b) Hence solve, in the interval 0 Ñ x Ñ 2π, the equation $\dfrac{\tan x - \sec x}{1 - \sin x} = \sqrt{2}$.

(3 marks)

11	Scheme	Marks	AOs	Pearson Progression Step and Progress descriptor
(a)	Writes tan$x$ and sec$x$ in terms of sin$x$ and cos$x$. For example, $\frac{\tan x - \sec x}{1 - \sin x} = \frac{\left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right)}{\left(\frac{1 - \sin x}{1}\right)}$	M1	2.1	5th Understand the functions sec, cosec and cot.
	Manipulates the expression to find $\left(\frac{\sin x - 1}{\cos x}\right) \times \left(\frac{1}{1 - \sin x}\right)$	M1	1.1b	
	Simplifies to find $-\frac{1}{\cos x} = -\sec x$	A1	1.1b	
		(3)		
(b)	States that $-\sec x = \sqrt{2}$ or $\sec x = -\sqrt{2}$	B1	2.2a	6th Use the functions sec, cosec and cot to solve simple trigonometric problems.
	Writes that $\cos x = -\frac{1}{\sqrt{2}}$ or $x = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$	M1	1.1b	
	Finds $x = \frac{3\pi}{4}, \frac{5\pi}{4}$	A1	1.1b	
		(3)		
				(6 marks)

Maths Genie Questions

Set 1 (A2)

1 Use proof by contradiction to show that there exist no integers x and y for which $6x + 9y = 1$.

(Total for question 1 is 4 marks)

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2 Use proof by contradiction to show that there exist no integers x and y for which $30x + 20y = 7$.

(Total for question 2 is 4 marks)

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3 Prove by contradiction that $\sqrt{3}$ is irrational

(Total for question 3 is 6 marks)

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4 Prove by contradiction that $\sqrt{2}$ is irrational

(Total for question 4 is 6 marks)

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5 Prove by contradiction that the sum of a rational number and an irrational number is irrational

(Total for question 5 is 6 marks)

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6 Prove by contradiction that there are infinitely many prime numbers

(Total for question 6 is 6 marks)

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1) Assume that x and y are integers and

$$6x + 9y = 1$$

$$3(2x + 3y) = 1$$

$$2x + 3y = \frac{1}{3}$$

if x and y are integers $2x$ and $3y$ must be integers.

Integer + Integer cannot $= \frac{1}{3}$

$\therefore$ assumption must be false, no integers ~~x and y~~ exist for which $6x + 9y = 1$

2) Assume x and y are integers and

$$30x + 20y = 7$$

$$10(3x + 2y) = 7$$

$$3x + 2y = \frac{7}{10}$$

$3x$ and $2y$ must be integers.

Integer + Integer = Integer $\therefore$ assumption must be incorrect.

$\therefore$ No integers exist for x and y for which $30x + 20y = 7$.

3/ assume $\sqrt{3}$ can be expressed in the form $\frac{a}{b}$ where a and b are integers with no common factor (except 1)

$$\sqrt{3} = \frac{a}{b}$$

$$3 = \frac{a^2}{b^2}$$

$$3b^2 = a^2$$

a^2 (and a) must be multiples of 3

$$3b^2 = (3n)^2$$
$$3b^2 = 9n^2$$
$$b^2 = 3n^2$$

b^2 (and b) must also be multiples of 3.

a and b have common factor 3. ∴ the assumption is incorrect and $\sqrt{3}$ is irrational

4/ Assume $\sqrt{2}$ can be expresred in the form $\frac{a}{b}$ where a and b are integers with no common factors except 1.

$$\sqrt{2} = \frac{a}{b}$$

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

a^2 (and a) must be even

$$2b^2 = (2n)^2$$
$$2b^2 = 4n^2$$

$$b^2 = 2n^2$$

b^2 (and b) must also be even

a and b have a common factor of 2. ∴ the assumption is incorrect and $\sqrt{2}$ is irrational

5/ Assume the sum of a rational number and an irrational number is rational

Let the rational number $= \frac{a}{b}$ (where a and b are integers)

Let the irrational number $= c$ (which cannot be expressed in the form $\frac{a}{b}$ where a and b are integers)

Let their sum $= \frac{d}{e}$ (where d and e are integers)

$$\frac{a}{b} + c = \frac{d}{e}$$

$$c = \frac{d}{e} - \frac{a}{b}$$

$$= \frac{bd}{be} - \frac{ae}{be}$$

$$= \frac{bd - ae}{be}$$

c has been written in the form $\frac{a}{b}$ where a and b are integers. ∴ the assumption is incorrect and the sum of a rational and irrational number must be irrational.

6) assume there are a finite number of primes.

Let x = the product of all prime numbers
$= P_1 \times P_2 \times P_3 \times \ldots \times P_n$

Let $y = x + 1$

y has no prime factors P_1 to P_n as there would always be a remainder of 1.

$\therefore$ either y is prime and the assumption is incorrect
or y is not prime and has a factor not listed and the assumption is incorrect.

Either way the assumption is incorrect and there are an infinite number of primes.

Set 2 (AS)

1 Prove that $x^2 - 4x + 7$ is positive for all values of x

(Total for question 1 is 3 marks)

2 Prove that $n^2 - n + 3$ is not a prime number for all values of n

(Total for question 2 is 2 marks)

3 Prove that the sum of two consecutive odd numbers is a multiple of 4

(Total for question 3 is 3 marks)

4 Prove that $(x + y)^2 \neq x^2 + y^2$

(Total for question 4 is 3 marks)

5 (a) Prove that $n^2 + n + 11$ is prime for all integers between 1 and 5. **(3)**

(b) Prove that $n^2 + n + 11$ is not prime for all values of n **(2)**

(Total for question 5 is 6 marks)

6 Prove by exhaustion that the sum of two even positive integers less than 10 is also even.

(Total for question 6 is 3 marks)

7 "If I multiply a number by 2 and add 5 the result is always greater than the original number."

State, giving a reason, whether the above statement is always true, sometimes true or never true.

(Total for question 7 is 2 marks)

8 Prove that $(2n + 3)^2 - (2n - 3)^2$ is a multiple of 6 for all values of n

(Total for question 8 is 4 marks)

9 Prove that the sum of the squares of two consecutive odd integers is always 2 more than a multiple of 8.

(Total for question 9 is 4 marks)

10 Prove that $n^2 + 7n + 15 > n + 3$ is true for all values of n

(Total for question 10 is 4 marks)

1) $x^2 - 4x + 7$

$(x-2)^2 - 4 + 7$

$(x-2)^2 + 3$

Min point (2, 3) ∴ positive for all values of x

2) $n^2 - n + 3$

when $n = 3$

$(3)^2 - 3 + 3 = 9$

9 is not prime $3 \times 3 = 9$ (disproof by counter example)

3) $2n + 1 + 2n + 3$

$4n + 4$

$4(n + 1)$

4) let $x = 1$ and $y = 2$

$(x + y)^2 = x^2 + y^2$

$(1 + 2)^2 = (1)^2 + (2)^2$

$3^2 = 1 + 4$

$9 = 1 + 4$

$9 = 5$

disproof by counter example.

5a) $n=1$ $(1)^2 + (1) + 11 = 13$ (prime)

$n=2$ $(2)^2 + (2) + 11 = 17$ (prime)

$n=3$ $(3)^2 + (3) + 11 = 23$ (prime)

$n=4$ $(4)^2 + (4) + 11 = 31$ (prime)

$n=5$ $(5)^2 + (5) + 11 = 41$ (prime)

proof by exhaustion.

b) $n=10$ $(10)^2 + 10 + 11 = 121$ (not prime $11 \times 11 = 121$

6) even positive integers less than 10 2, 4, 6, 8

$2 + 4 = 6$ (even)

$2 + 6 = 8$ (even)

$2 + 8 = 10$ (even)

$4 + 6 = 10$ (even)

$4 + 8 = 12$ (even)

$6 + 8 = 14$ (even)

7) $2n + 5$

when $n = 10$ $2(10) + 5 = 25$ (True)

when $n = -10$ $2(-10) + 5 = -15$ (Not True)

The statement is sometimes true.

8) $(2n+3)^2 - (2n-3)^2$

$(2n+3)(2n+3) - (2n-3)(2n-3)$

$(4n^2+6n+6n+9) - (4n^2-6n-6n+9)$

$(4n^2+12n+9) - (4n^2-12n+9)$

$4n^2+12n+9-4n^2+12n-9$

$24n$

$\underline{\underline{6(4n)}}$

9) $(2n+1)^2 + (2n+3)^2$

$(2n+1)(2n+1) + (2n+3)(2n+3)$

$4n^2+2n+2n+1 + 4n^2+6n+6n+9$

$8n^2+16n+10$

$8n^2+16n+8+2$

$\underline{8(n^2+2n+1)+2}$

10) $n^2+7n+15 > n+3$

$n^2+6n+12 > 0$

$(n+3)^2-9+12 > 0$

$(n+3)^2+3 > 0$

$(n+3)^2+3$ has a minimum value when $n=-3$

min value $= 3$

$3 > 0$

$\therefore\ n^2+7n+15 > n+3$ for all values of n