Mathematical Analysis 2017-8

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Tutorial problems 4 - Sequences

Question 1.

Prove the following using $\epsilon - N$ (ie. from first principles, don't assume any other results about convergent sequences);

1. The constant sequence (k_n), with $k_n = k$ for some $k \in \mathbb{R}$, converges to k.
2. Let $a_n = \frac{1}{n^2}$. Then $a_n \to 0$ as $n \to \infty$.
3. Let $b_n = n^2$. Then the sequence (b_n) does not converge.
4. Let $c_n \to c$ and $d_n \to d$ be convergent sequences. Let $x_n = 2c_n - 3d_n$. Then $x_n \to 2c - 3d$ as $n \to \infty$.
5. Let $w > 1$ and $u_n = w^{\frac{1}{n}}$. Then $u_n \to 1$ as $n \to \infty$.
6. Let $0 < z < 1$ and $v_n = z^{\frac{1}{n}}$. Then $v_n \to 1$ as $n \to \infty$.

Answer:

Claim: Let $k_n = k$. Then $k_n \to k$ as $n \to \infty$.

Proof. Let $\epsilon > 0$. Choose *any* $N \in \mathbb{N}^+$. Then,

$$|k_n - k| = 0 < \epsilon$$

□

Claim: Let $a_n = \frac{1}{n^2}$. Then $a_n \to 0$ as $n \to \infty$.

Rough work:

$$\left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \epsilon$$

$$n > \frac{1}{\sqrt{\epsilon}}$$

End of rough work.

Proof. Let $\epsilon > 0$. Choose an $N \in \mathbb{N}^+$ such that $N > \frac{1}{\sqrt{\epsilon}}$. Then for $n > N$, then $n > \frac{1}{\sqrt{\epsilon}}$ so that, $\frac{1}{n^2} < \epsilon$

$$|a_n - 0| = \frac{1}{n^2} < \epsilon$$

□

Claim: Let $b_n = n^2$. Then (b_n) does not converge.

Proof. Assume for contradiction that $b_n \to b$ for some $b \in \mathbb{R}$.

Then there exists $N \in \mathbb{N}^+$ such that for all $n > N$,

$$|b_n - b| = \left|n^2 - b\right| < 1$$

However, consider $n > \sqrt{1 + |b|}$. Then $n^2 - |b| > 1$, and so, $n^2 - b > 1$ and hence,

$$\left|n^2 - b\right| > 1$$

but this is a contradiction.

□

Claim: Let $x_n = 2c_n - 3d_n$ with $c_n \to c$, $d_n \to d$ as $n \to \infty$. Then $x_n \to 2c - 3d$ as $n \to \infty$.

Rough work:

$$|x_n - (2c - 3d)| = |2(c_n - c) - 3(d_n - d)| \leq |2(c_n - c)| + |3(d_n - d)| = 2\,|c_n - c| + 3\,|d_n - d|$$

End of rough work.

Proof. Let $\epsilon > 0$.

There exists $N_1 \in \mathbb{N}^+$ such that $|c_n - c| < \frac{\epsilon}{4}$ for $n > N_1$.

There exists $N_2 \in \mathbb{N}^+$ such that $|d_n - d| < \frac{\epsilon}{6}$ for $n > N_2$.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$|x_n - (2c - 3d)| \leq 2\,|c_n - c| + 3\,|d_n - d| < 2 \cdot \frac{\epsilon}{4} + 3 \cdot \frac{\epsilon}{6} = \epsilon$$

□

Claim: Let $w > 1$ and $u_n = w^{\frac{1}{n}}$. Then $u_n \to 1$ as $n \to \infty$.

Rough work:

Note that $w^{\frac{1}{n}} > 1$ for $w > 1$.

$$\begin{aligned} w^{\frac{1}{n}} - 1 &< \epsilon \\ w^{\frac{1}{n}} &< 1 + \epsilon \\ \frac{1}{n} \log w &< \log(1+\epsilon) \\ n &> \frac{\log w}{\log(1+\epsilon)} \end{aligned}$$

End of rough work.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}^+$ such that $N > \frac{\log w}{\log(1+\epsilon)}$. Then for $n > N$ we have, $n > \frac{\log w}{\log(1+\epsilon)}$ so, $\frac{1}{n} \log w < \log(1+\epsilon)$, and so $w^{\frac{1}{n}} - 1 < \epsilon$. Noting that $w^{\frac{1}{n}} > 1$, then for $n > N$ we have,

$$|u_n - 1| = \left| w^{\frac{1}{n}} - 1 \right| = w^{\frac{1}{n}} - 1 < \epsilon$$

$\square$

Claim: Let $0 < z < 1$ and $v_n = z^{\frac{1}{n}}$. Then $v_n \to 1$ as $n \to \infty$.

Rough work:

Note that $z^{\frac{1}{n}} < 1$ for $0 < z < 1$. Assume that $\epsilon < 1$ otherwise the first inequality below is trivial.

$$\begin{aligned} 1 - z^{\frac{1}{n}} &< \epsilon \\ z^{\frac{1}{n}} &> 1 - \epsilon \\ \frac{1}{n} \log z &> \log(1-\epsilon) \\ \frac{1}{n} &< \frac{\log(1-\epsilon)}{\log z} \\ n &> \frac{\log z}{\log(1-\epsilon)} \end{aligned}$$

where we note $\log z < 0$, and $\log(1-\epsilon) < 0$.

End of rough work.

Proof. Let $\epsilon > 0$.

If $\epsilon < 1$ choose $N \in \mathbb{N}^+$ such that $N > \frac{\log z}{\log(1-\epsilon)}$.

If $\epsilon \geq 1$, then (for any n),

$$|v_n - 1| = \left|1 - z^{\frac{1}{n}}\right| = 1 - z^{\frac{1}{n}} < 1 \leq \epsilon$$

Otherwise $\epsilon < 1$, so for $n > N$ we have, $n > \frac{\log z}{\log(1-\epsilon)}$ so, $\frac{1}{n} \log z > \log(1-\epsilon)$ (recalling that $\log z < 0$), and so $z^{\frac{1}{n}} > 1 - \epsilon$. Then, for $n > N$ we have,

$$|v_n - 1| = \left|1 - z^{\frac{1}{n}}\right| = 1 - z^{\frac{1}{n}} < \epsilon$$

□

Question 2.

Prove that the sequence, (x_n), defined by,

$$x_{n+1} = \frac{3x_n - x_{n-1}}{2}\ , \quad n \geq 2$$

converges for any real values of x_1, x_2.

[Hint: show it is a Cauchy sequence.]

Answer:

Claim: x_n defined above is a Cauchy sequence.

Proof. From the definition we see;

$$x_{n+1} - x_n = \frac{x_n - x_{n-1}}{2}$$

so that for $n > 1$,

$$x_{n+1} - x_n = \frac{x_2 - x_1}{2^{n-1}}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}^+$ such that $N > \log_2\left(\frac{4|x_2-x_1|}{\epsilon}\right)$ ie. then, $\frac{|x_2-x_1|}{2^{N-2}} < \epsilon$.

Then for all $n > m > N$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - \ldots - x_{m+1} + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{m+1} - x_m| \\
&= \frac{|x_2 - x_1|}{2^{n-2}} + \frac{|x_2 - x_1|}{2^{n-3}} + \ldots + \frac{|x_2 - x_1|}{2^{m-1}} \\
&< \frac{|x_2 - x_1|}{2^{m-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right) \\
&= \frac{|x_2 - x_1|}{2^{m-1}}\frac{1}{1-\frac{1}{2}} = \frac{|x_2 - x_1|}{2^{m-2}} < \frac{|x_2 - x_1|}{2^{N-2}} < \epsilon
\end{aligned}$$

Hence it is a Cauchy sequence (and therefore converges).

□

Question 3.

Prove that the sequence, (x_n), defined by,

$$x_{n+1} = \arctan x_n \ , \quad n > 1$$

converges for any $x_1 \in \mathbb{R}$.

[Hint: Sketch a graph of arctan to figure out how this sequence behaves. You may use the fact that $\arctan x < x$ for $x > 0$.]

Answer:

Claim: (x_n) defined above converges.

Proof. There are 3 cases;

- $x_1 = 0$. In this case $x_n = 0$, since $\arctan 0 = 0$. Hence the sequence converges to zero.
- $x_1 > 0$. Since $\arctan x > 0$ for $x > 0$ then all $x_n > 0$. Hence (x_n) is bounded from below by zero. Then (x_n) is decreasing, since $x_{n+1} = \arctan x_n < x_n$ (for $x_n > 0$). Since (x_n) is bounded below and decreasing it is convergent.
- $x_1 < 0$. Since $\arctan x < 0$ for $x < 0$ then all $x_n < 0$. Hence (x_n) is bounded from above by zero. Then (x_n) is increasing, since $x_{n+1} = \arctan x_n > x_n$ (for $x_n < 0$). Since (x_n) is bounded above and increasing it is convergent.

$\square$