Mathematical Analysis 2017-8

Toby Wiseman

Tutorial problems 5 - Series

Question 1.

Carefully prove whether the following series are convergent or divergent;

a) $\sum_{k=1}^{\infty} \frac{k^2-1}{k^2+k+1}$

b) $\sum_{k=1}^{\infty} (-1)^k$

c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

d) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$

e) $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

f) $\sum_{k=1}^{\infty} \frac{k+1}{k!}$

g) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{1+k^2}}$

You may use the standard tests (eg. comparison, alternating series, root, ratio) without proof and also assume;

- the geometric series $\sum_{k=0}^{\infty} x^k$ is convergent for $|x| < 1$.
- the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p = 1$.

Question 2.

Show that $\sum_{k=1}^{\infty} \frac{b_k}{k^2}$ converges, where (b_k) is a sequence where $|b_k| < B$ for $B \in \mathbb{R}$ and all $k \in \mathbb{N}^+$.

Question 3.

Use the Ratio test to prove the following;

(a) the Taylor series (about zero) of the exponential function, the power series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, converges **absolutely** for all $x \in \mathbb{R}$.

(b) the Taylor series of the logarithm (about one) given by the power series $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$, converges **absolutely** for $|x| < 1$ and diverges for $|x| > 1$.
What happens for $x = \pm 1$?