Mathematical Analysis 2017-8

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Tutorial problems 5 - Series

Question 1.

Carefully prove whether the following series are convergent or divergent;

a) $\sum_{k=1}^{\infty} \frac{k^2-1}{k^2+k+1}$

b) $\sum_{k=1}^{\infty} (-1)^k$

c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

d) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$

e) $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

f) $\sum_{k=1}^{\infty} \frac{k+1}{k!}$

g) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{1+k^2}}$

You may use the standard tests (eg. comparison, alternating series, root, ratio) without proof and also assume;

- the geometric series $\sum_{k=0}^{\infty} x^k$ is convergent for $|x| < 1$.
- the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p = 1$.

Answer:

For these answers we recall the comparison test results;

- Let $\sum_{k=1}^{\infty} b_k$ be a convergent series such that $0 \leq b_k$. Then the series $\sum_{k=1}^{\infty} a_k$ converges if there exists $N \in \mathbb{N}^+$ such that $|a_k| \leq b_k$ for all $k > N$.
- Let $\sum_{k=1}^{\infty} b_k$ be a divergent series such that $0 \leq b_k$. Then the series $\sum_{k=1}^{\infty} a_k$ diverges if there exists $N \in \mathbb{N}^+$ such that $0 \leq b_k \leq a_k$ for all $k > N$.

Also remember the Cauchy convergence test has the corollary that a necessary (but not sufficient) condition for $\sum_{k=1}^{\infty} b_k$ to converge is that $b_k \to 0$ as $k \to \infty$.

Claim: $\sum_{k=1}^{\infty} \frac{k^2-1}{k^2+k+1}$ diverges.

Proof. This series is $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = \frac{k^2 - 1}{k^2 + k + 1}$$

Then $a_k \to 1$ as $k \to \infty$. A standard result (the Corollary of the Cauchy convergence test in the lecture notes) is that a necessary condition for convergence is that $a_k \to 0$, so this series diverges.

□

Claim: $\sum_{k=1}^{\infty} (-1)^k$ diverges.

Proof. This series is $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = (-1)^k$$

Then $a_k \to 1$ as $k \to \infty$. Since a_k does not tend to zero, for the same reason as above, this implies the series diverges.

□

Claim: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

Proof. Consider the series $\sum_{k=1}^{\infty} a_k$, where,

$$a_k = \frac{1}{\sqrt{k}} \geq \frac{1}{k}$$

for all $k \in \mathbb{N}^+$.

Now $\sum_{k=1}^{\infty} b_k$ with $b_k = \frac{1}{k}$ diverges, and $0 \leq b_k \leq a_k$ and by the comparison test hence $\sum_{k=1}^{\infty} a_k$ diverges.

□

Claim: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$ converges.

Proof. This series is $\sum_{k=1}^{\infty} (-1)^{k-1} b_k$, where,

$$b_k = \frac{1}{\sqrt{k}}$$

Note that (b_k) is a decreasing sequence with $b_k \to 0$ as $k \to \infty$. By the alternating series test then the series converges.

□

Claim: $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges.

Proof. The series is $\sum_{k=1}^{\infty} a_k$, where,

$$a_k = \frac{1}{1+k^2} \leq \frac{1}{k^2}$$

Then we have $|a_k| \leq b_k = \frac{1}{k^2}$. Since $\sum_{k=1}^{\infty} b_k$ is convergent, then by the comparison test $\sum_{k=1}^{\infty} a_k$ is convergent.

□

Claim: $\sum_{k=1}^{\infty} \frac{k+1}{k!}$ converges.

Proof. The series is $\sum_{k=1}^{\infty} a_k$, where,

$$a_k = \frac{k+1}{k!}$$

Consider the ratio test, so,

$$y_k = \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{k+2}{(k+1)!}}{\frac{k+1}{k!}}\right| = \frac{k+2}{k+1}\frac{1}{(k+1)}$$

We see $y_k \to 0$ as $k \to \infty$ and hence by the ratio test the series converges.

□

Claim: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{1+k^2}}$ diverges.

Proof. The series is $\sum_{k=1}^{\infty} a_k$, where,

$$a_k = \frac{1}{\sqrt{1+k^2}} = \frac{1}{k}\frac{1}{\sqrt{1+\frac{1}{k^2}}}$$

Since $\frac{1}{\sqrt{1+\frac{1}{k^2}}} \to 1$ as $k \to \infty$, then we may find N such that for $k > N$ then,

$$a_k > \frac{1}{2}\frac{1}{k}$$

Then we have $0 < b_k = \frac{1}{2k} < a_k$. Since $\sum_{k=1}^{\infty} b_k$ is divergent, then by the comparison test $\sum_{k=1}^{\infty} a_k$ is divergent.

□

Question 2.

Show that $\sum_{k=1}^{\infty} \frac{b_k}{k^2}$ converges, where (b_k) is a sequence where $|b_k| < B$ for $B \in \mathbb{R}$ and all $k \in \mathbb{N}^+$.

Answer:

Claim: $\sum_{k=1}^{\infty} \frac{b_k}{k^2}$ converges for $|b_k| < B$ for $B \in \mathbb{R}$.

Proof. Write the sum as $\sum_{k=1}^{\infty} a_k$ with,

$$a_k = \frac{b_k}{k^2}$$

so that,

$$|a_k| = \frac{|b_k|}{k^2} \leq \frac{B}{k^2}$$

Now since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does $\sum_{k=1}^{\infty} \frac{B}{k^2}$, so by the comparison test our sum $\sum_{k=1}^{\infty} a_k$ converges too since $|a_k| \leq \frac{B}{k^2}$.

□

Question 3.

Use the Ratio test to prove the following;

(a) the Taylor series (about zero) of the exponential function, the power series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, converges **absolutely** for all $x \in \mathbb{R}$.

(b) the Taylor series of the logarithm (about one) given by the power series $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$, converges **absolutely** for $|x| < 1$ and diverges for $|x| > 1$.
What happens for $x = \pm 1$?

Answer:

Claim: $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges absolutely for any $x \in \mathbb{R}$.

Proof. We must show that $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges absolutely, and hence must show $\sum_{n=0}^{\infty} \frac{1}{n!} |x|^n$ converges.

Write the sum as $\sum_{n=0}^{\infty} a_n$ with,

$$a_n = \frac{1}{n!} |x|^n$$

Consider the Ratio test, so,

$$y_n = \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+1)!} |x|^{n+1}}{\frac{1}{n!} |x|^n} \right| = \frac{1}{(n+1)} |x|$$

For any x this has the limit, $y_n \to 0$ as $n \to \infty$. Hence by the Ratio test the sum converges. □

Claim: $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ converges absolutely for any $|x| < 1$.

Proof. We must show that $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ converges absolutely for any $|x| < 1$, and hence must show $\sum_{n=1}^{\infty} \frac{1}{n} |x|^n$ converges.

Write the sum as $\sum_{n=1}^{\infty} a_n$ with,

$$a_n = \frac{1}{n} |x|^n$$

Consider the Ratio test, so,

$$y_n = \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{(n+1)} |x|^{n+1}}{\frac{1}{n} |x|^n} \right| = \frac{n}{(n+1)} |x|$$

Then since $\frac{n}{(n+1)} \to 1$ we have, $y_n \to |x|$ as $n \to \infty$. Hence by the ratio test the series converges for $|x| < 1$ and diverges for $|x| > 1$. It is inconclusive for $|x| = 1$. □

Claim: $\sum_{n=1}^{\infty} \frac{1}{n}x^n$ diverges for $x = +1$.

Proof. When $x = +1$ we have the series,

$$-\sum_{n=1}^{\infty} \frac{1}{n}(+1)^n = -\sum_{n=1}^{\infty} \frac{1}{n}$$

which is divergent.

□

Claim: $\sum_{n=1}^{\infty} \frac{1}{n}x^n$ converges for $x = -1$ (but not absolutely).

Proof. When $x = -1$ we have,

$$\ln(2) = -\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n-1}$$

and since this is an alternating series of the form $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ with $b_k \geq 0$, by the alternating series test this converges (since $b_k = 1/k \to 0$ as $k \to \infty$).

However, it does not converge absolutely since then $\sum_{n=1}^{\infty} \left|\frac{1}{n}(-1)^n\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

□